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SECOND PERIOD-DOUBLING IN A THREE-DIMENSIONAL SYSTEM

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Introduction

The purpose of this work is to derive an analytical approximation of the critical parameter value corresponding to the second period-doubling bifurcation (period-two to period-four bifurcation) in the three-dimensional autonomous system

$$
\begin{aligned}
\dot{x} &= \mu x - y - xz, \\
\dot{y} &= \mu y + x, \\
\dot{z} &= -z + x^2 z + y^2,
\end{aligned}
\tag{1}
$$

where x, y, z are scalar variables and μ the control parameter of the system. The origin of equations (1) is a stable equilibrium for $\mu < 0$ and unstable for $\mu > 0$ so that a Hopf bifurcation occurs at $\mu = 0$. As μ increases from zero, the periodic orbit undergoes a symmetry-breaking bifurcation at $\mu = \mu_{SB}$, and the first and the second period-doublings at $\mu = \mu_{PD1}$ and $\mu = \mu_{PD2}$, respectively. As μ increases again complicated dynamics takes place in the system.

The system (1) has been investigated recently by several authors. Rand [7] used the center manifold theory to construct a first-order approximation of the periodic orbit near the Hopf bifurcation. An approximation to the critical value $\mu_{PD1}(\approx 0.45)$ was then calculated by performing a stability analysis of the orbit. Nayfeh and Balachandran [5] used the multiple scales technique to obtain the same first-order approximation of the periodic orbit than Rand [7]. The approximations $\mu_{SB}(\approx 0.31)$ and $\mu_{PD1}(\approx 0.446)$ to the critical values were obtained numerically using the Floquet theory [6]. Recently a higher-order expansion of the periodic orbit using the multiple scales method has been derived [1]. Using this higher-order approximation, the stability analysis was achieved to predict the analytical approximations of the symmetry-breaking bifurcation $\mu_{SB}(\approx 0.31)$ and to improve the approximation of the critical value $\mu_{PD1}(\approx 0.446)$.

In this work we derive an analytical approximation of the critical parameter value μ_{PD2} corresponding to the period-two to period-four bifurcation point. Comparison with numerical simulation is also provided.

Second period-doubling

Using the method of multiple scales [4], a higher-order asymptotic expansion for the periodic orbit of the system (1) may be sought in the form

$$\begin{aligned} x &= \sum_{n=0}^{6} \varepsilon^n x_n(T_0,T_1,T_2,T_3,T_4,T_5) + \cdots, \\ y &= \sum_{n=0}^{6} \varepsilon^n y_n(T_0,T_1,T_2,T_3,T_4,T_5) + \cdots, \\ z &= \sum_{n=0}^{6} \varepsilon^n z_n(T_0,T_1,T_2,T_3,T_4,T_5) + \cdots, \end{aligned} \tag{2}$$

where $T_n = \varepsilon^n t$ are the time scales and ε is a small parameter. The control parameter is expanded as $\mu = \varepsilon^2\mu_2 + O(\varepsilon^3)$. Substituting this last relation and Equations (2) into (1) and equating coefficients of like powers of ε, we obtain at different orders of ε the following systems of successive approximations x_n, y_n, z_n

$$o(\varepsilon^1): \qquad \begin{aligned} D_0x_1 + y_1 &= 0, \\ D_0y_1 - x_1 &= 0, \\ D_0z_1 + z_1 &= 0. \end{aligned} \tag{3}$$

$$o(\varepsilon^i, i \geq 2): \qquad \begin{aligned} D_0x_i + y_i &= \mu_2 x_{i-2} - \sum_{j=0}^{i} x_j z_{i-j} - \sum_{j=1}^{i} D_j x_{i-j}, \\ D_0y_i - x_i &= \mu_2 y_{i-2} - \sum_{j=1}^{i} D_j y_{i-j}, \\ D_0z_i + z_i &= \sum_{j=0}^{i} y_j y_{i-j} - \sum_{j=1}^{i} D_j z_{i-j} + \sum_{j=0}^{i}\left(x_{i-j}\sum_{k=0}^{j} z_k x_{j-k}\right), \end{aligned} \tag{4}$$

where $D_n = \dfrac{\partial}{\partial T_n}$. For details see [1].

In the analysis followed in this work to calculate an approximation of the second period-doubling value μ_{PD2}, we postulate the solution of the system (3) as

$$\begin{aligned} x_1 &= i\left[A_1 e^{i\frac{T_0}{q}} + (1-\delta_{q,1})A_2 e^{2i\frac{T_0}{q}} + (1-\delta_{q,1})A_3 e^{3i\frac{T_0}{q}}\right] + c.c, \\ y_1 &= \frac{1}{q}\left[A_1 e^{i\frac{T_0}{q}} + 2(1-\delta_{q,1})A_2 e^{2i\frac{T_0}{q}} + 3(1-\delta_{q,1})A_3 e^{3i\frac{T_0}{q}}\right] + c.c, \\ z_1 &= 0, \end{aligned} \tag{5}$$

where q is an integer and $\delta_{i,j}$ is the Kronecker symbol. In the particular case $q = 1$, we recover the previous results given in [1]. The introduction of the subharmonic terms at the first order of the approximation of the solution allows an analysis bifurcation near the second period-doubling branch (see Figure 1).

Following the same procedure as in [1], we obtain the set of conditions which vanish the secular terms as follows

$$D_1A_1 = 0 \quad \text{and} \quad (1-\delta_{q,1})D_1A_j = 0, \qquad (j = 2,3), \tag{6}$$

$$\begin{aligned}
&\frac{2}{q}D_2A_1 - \frac{2}{q}\mu_2A_1 + \frac{1}{q^3}\left[\left(2 - \frac{1}{1+\frac{2i}{q}} + \right)\overline{A_1}A_1^2\right. \\
&+(1-\delta_{q,1})^2\left(8 + \frac{4}{1-\frac{i}{q}} - \frac{4}{1+\frac{3i}{q}}\right)\overline{A_2}A_2A_1 + (1-\delta_{q,1})\left(\frac{1}{1-\frac{2i}{q}} - \frac{6}{1+\frac{2i}{q}}\right)\overline{A_1}^2A_3 \\
&\left.+(1-\delta_{q,1})^2\left(18 + \frac{6}{1-\frac{2i}{q}} - \frac{6}{1+\frac{4i}{q}}\right)\overline{A_3}A_3A_1 + (1-\delta_{q,1})^3\left(\frac{3}{1-\frac{i}{q}} - \frac{4}{1+\frac{4i}{q}}\right)\overline{A_3}A_2^2\right] = 0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\frac{4(1-\delta_{q,1})}{q}D_2A_2 - \frac{4(1-\delta_{q,1})}{q}\mu_2A_2 + \frac{2}{q^3}\left[(1-\delta_{q,1})^3\left(8 - \frac{4}{1+\frac{4i}{q}}\right)\overline{A_2}A_2^2\right. \\
&+(1-\delta_{q,1})\left(2 + \frac{4}{1+\frac{i}{q}} - \frac{4}{1+\frac{3i}{q}}\right)\overline{A_1}A_1A_2 + (1-\delta_{q,1})^3\left(18 + \frac{3}{1-\frac{i}{q}} - \frac{12}{1+\frac{5i}{q}}\right)\overline{A_3}A_3A_2 \\
&\left.+(1-\delta_{q,1})^2\left(\frac{3}{1+\frac{i}{q}} - \frac{6}{1+\frac{4i}{q}} + \frac{4}{1-\frac{i}{q}}\right)\overline{A_2}A_1A_3\right] = 0,
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\frac{6(1-\delta_{q,1})}{q}D_2A_3 - \frac{6(1-\delta_{q,1})}{q}\mu_2A_3 + \frac{3}{q^3}\left[\frac{(1-\delta_{q,1})^3}{1+\frac{2i}{q}}A_1^3\right. \\
&+(1-\delta_{q,1})\left(2 + \frac{6}{1+\frac{2i}{q}} - \frac{6}{1+\frac{4i}{q}}\right)\overline{A_1}A_1A_3 + (1-\delta_{q,1})^3\left(18 - \frac{9}{1+\frac{6i}{q}}\right)\overline{A_3}A_3^2 \\
&\left.+(1-\delta_{q,1})^2\left(4 + \frac{3}{1+\frac{i}{q}} - \frac{12}{1+\frac{5i}{q}}\right)\overline{A_2}A_2A_3 + (1-\delta_{q,1})^2\left(\frac{4}{1+\frac{i}{q}} - \frac{4}{1+\frac{4i}{q}}\right)\overline{A_1}A_2^2\right] = 0.
\end{aligned} \tag{9}$$

Substituting $A_j(T_1,T_2) = \frac{1}{2}a_j(T_1,T_2)e^{i\beta_j(T_1,T_2)}$ (where a and β are real quantities) into Equations (7)-(9), separating real and imaginary parts, we obtain for $q=1$ the same amplitude equation that in [1].

For $q=2$, the set of Equations (6)-(9) reduces to the following system

$$\begin{aligned}
\frac{da_1}{dT_2} &= \mu_2 a_1 - \frac{3}{64}a_1^3 - \frac{81}{260}a_2^2 a_1 - \frac{99}{160}a_3^2 a_1 - \frac{1}{20}a_2^2 a_3 + \frac{5}{64}a_1^2 a_3, \\
\frac{da_2}{dT_2} &= \mu_2 a_2 - \frac{9}{40}a_2^3 - \frac{129}{1040}a_1^2 a_2 - \frac{11}{80}a_1 a_2 a_3 - \frac{1359}{2320}a_3^2 a_2, \\
\frac{da_3}{dT_2} &= \mu_2 a_3 - \frac{171}{320}a_3^3 - \frac{3}{40}a_2^2 a_1 - \frac{19}{160}a_1^2 a_3 - \frac{172}{1160}a_2^2 a_3 - \frac{1}{64}a_1^3.
\end{aligned} \tag{10}$$

Solving Equations (10) we obtain an approximation of the three amplitudes as follows

$$\begin{aligned}
a_1 &= \frac{10^3}{3}\sqrt{\frac{26}{2126847}\mu}, \\
a_2 &= \frac{63}{50}a_1, \\
a_3 &= \frac{14}{25}a_1.
\end{aligned}$$

The stability analysis [1] leads here to the two equations

$$-1 + \frac{20}{9}\mu_{PD1} + \frac{25}{324}\mu_{PD1}^3 = 0, \tag{11}$$

$$-1 + \frac{4714450}{2392703}\mu_{PD2} + \frac{5136947}{25047448}\mu_{PD2}^3 = 0. \tag{12}$$

Solving the Equations (11) and (12), corresponding to $q=1$ and $q=2$, respectively, we obtain an approximation of the first and the second period-doubling bifurcations $\mu_{PD1} = 0.446$ and $\mu_{PD2} = 0.486$, simultaneously. To compare these critical parameter values with numerical calculations, see next section.

Numerical Study

In this section we describe the dynamical behaviour found numerically in system (1) in the vicinity of the two first period-doubling bifurcations. To do this, we have used the software continuation code AUTO94 [2].

The stability analysis of the Hopf bifurcation (see, for instance, [3]) reveals that it is supercritical: a stable symmetric periodic orbit emerges for $\mu > 0$. The evolution of this periodic orbit is schematized in the bifurcation diagram of Figure 1. In this qualitative figure we have indicated the Hopf bifurcation by an empty square, the symmetry-breaking bifurcation by an inverted triangle and the period-doubling bifurcation by a filled circle. Note that we represent the period of the T, $2T$ and $4T$-orbits divided by 1, 2 and 4, respectively.

First, the periodic orbit exhibits a symmetry-breaking bifurcation, SB ($\mu = 0.3150232$), to become a saddle orbit and a pair of asymmetric stable periodic orbits emerges.

Now we focus on the pair of asymmetric stable periodic orbits emerged at SB. These orbits become saddle when they exhibit a period-doubling bifurcation, PD1 ($\mu = 0.4403559$). Note that this flip bifurcation was analytically predicted to occur for $\mu_{PD1} = 0.446$ [1]. From

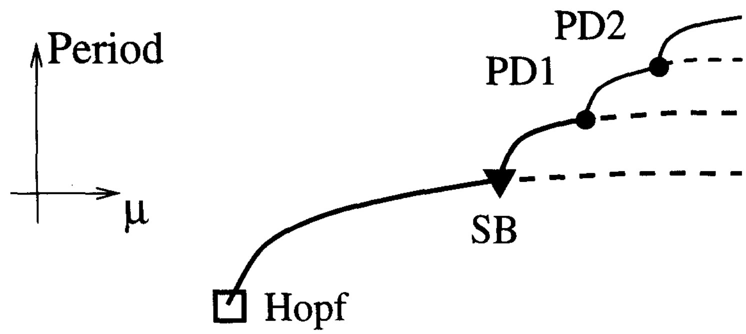

Figure 1: Partial bifurcation diagram of the periodic orbit emerged from the Hopf bifurcation. In this qualitative figure the solid line means stable periodic orbit and the dashed line saddle periodic orbit.

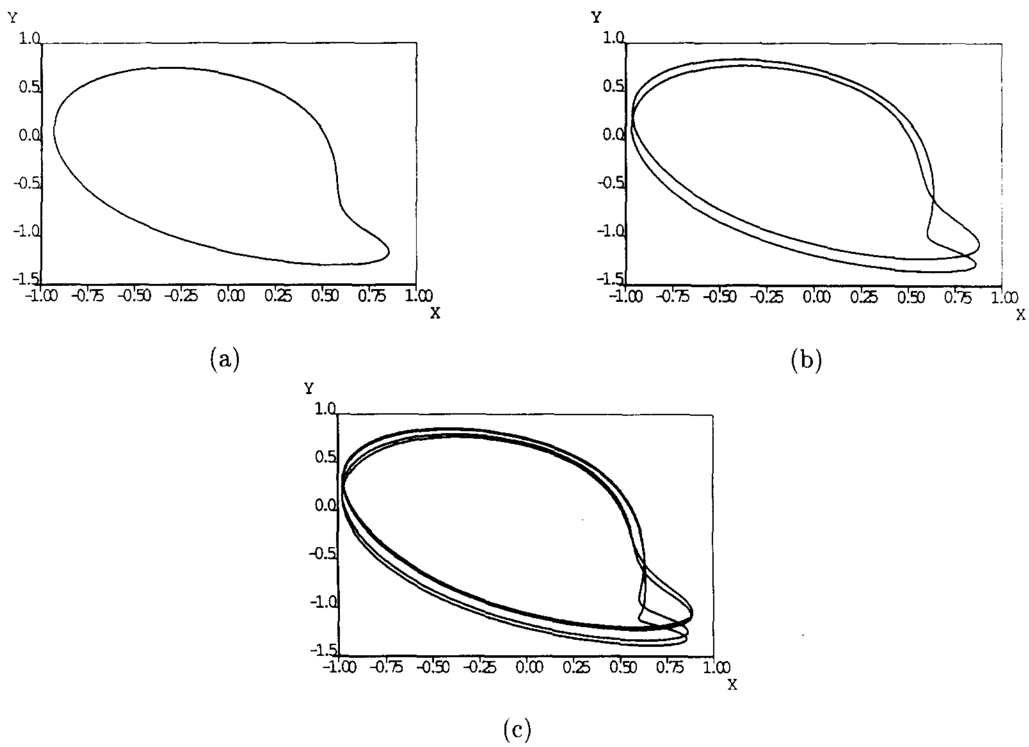

Figure 2: (a) Asymmetric periodic orbit, that emerged from SB, just at the flip bifurcation PD1 ($\mu = 0.4403559$). (b) $2T$-orbit at the point PD2 ($\mu = 0.4765392$). (c) A stable $4T$-orbit for $\mu = 0.485$.

such a bifurcation point a stable periodic orbit of approximately twice the period ($2T$-orbit) of the original orbit emerges.

The asymmetric $2T$-orbit (in fact a pair, due to the symmetry the system has) born at PD1 becomes non-stable in a flip bifurcation PD2 ($\mu = 0.4765392$) where a $4T$-orbit emerges. The analysis developed in this work predicted this period-doubling bifurcation to occur for $\mu_{PD2} = 0.486$.

Finally, we show, in Figure 2(a), the asymmetric periodic orbit, that emerged from SB, just at the point where it exhibits the flip bifurcation PD1. In Figure 2(b) we have represented the $2T$-orbit at the point PD2. A stable $4T$-orbit is showed in Figure 2(c), for $\mu = 0.485$.

Conclusions

The three-dimensional system considered in this paper has been investigated in terms of bifurcations in several papers [7, 5, 1]. Attention was focused principally on the symmetry-breaking and on the first period-doubling bifurcations following the Hopf bifurcation. In the present work we have studied the second period-doubling bifurcation. By introducing a suitable form of periodic solution at the first-order multiple scale expansion, we have derived two algebraic equations leading to an approximation of the first and the second period-doubling (period-one to period-two and period-two to period-four) bifurcations. The analytical procedure developed here provides these two bifurcations simultaneously. The result given in [1] may be considered as a particular case of the one given here. For validating the analytical prediction regarding the critical value μ_{PD2} numerical simulation was carried out and a good agreement was found.

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