The $\{-3\}$-reconstruction and the $\{-3\}$-self duality of tournaments

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Abstract

Let $T = (V, A)$ be a (finite) tournament and k be a non negative integer. For every subset X of V is associated the subtournament $T[X] = (X, A \cap (X \times X))$ of T, induced by X. The dual tournament of T, denoted by T^*, is the tournament obtained from T by reversing all its arcs. The tournament T is self dual if it is isomorphic to its dual. T is $\{-k\}$-self dual if for each set X of k vertices, $T[V \setminus X]$ is self dual. T is strongly self dual if each of its induced subtournaments is self dual. A subset I of V is an interval of T if for $a, b \in I$ and for $x \in V \setminus I$, $(a, x) \in A$ if and only if $(b, x) \in A$. For instance, $\emptyset$, V and $\{x\}$, where $x \in V$, are intervals of T called trivial intervals. T is indecomposable if all its intervals are trivial; otherwise, it is decomposable. A tournament T', on the set V, is $\{-k\}$-hypomorphic to T if for each set X on k vertices, $T[V \setminus X]$ and $T'[V \setminus X]$ are isomorphic. The tournament T is $\{-k\}$-reconstructible if each tournament $\{-k\}$-hypomorphic to T is isomorphic to it.

Suppose that T is decomposable and $\mid V \mid \geq 9$. In this paper, we begin by proving the equivalence between the $\{-3\}$-self duality and the strong self duality of T. Then we characterize each tournament $\{-3\}$-hypomorphic to T. As a consequence of this characterization, we prove that if there is no interval X of T such that $T[X]$ is indecomposable and $\mid V \setminus X \mid \leq 2$, then T is $\{-3\}$-reconstructible. Finally, we conclude by reducing the $\{-3\}$-reconstruction problem

to the indecomposable case (between a tournament and its dual). In particular, we find and improve, in a less complicated way, the results of [6] found by Y. Boudabbous and A. Boussaïri.

1 Introduction

1.1 Preliminaries on tournaments

A (finite) *tournament* T consists of a finite set V of *vertices* with a prescribed collection A of ordered pairs of distinct vertices, called the set of *arcs* of T, which satisfies: for $x, y \in V$ with $x \neq y$, $(x, y) \in A$ if and only if $(y, x) \notin A$. Such a tournament is denoted by (V, A). If (x, y) is an arc of T, then we say that x *dominates* y (symbolically $x \rightarrow y$). The *dual* of the tournament T is the tournament $T^* = (V, A^*)$ defined by: for all x, $y \in V$, $(y, x) \in A^*$ if and only if $(x, y) \in A$. The tournament T is *transitive* or a *linear order* provided that for any $x, y, z \in V$, if $(x, y) \in A$ and $(y, z) \in A$, then $(x, z) \in A$. For example, a total order on a finite set E can be identified to a transitive tournament with a vertex set E in the following way: for $x, y \in E$ with $x \neq y$, $x \rightarrow y$ if and only if $x < y$. The tournament corresponding to the usual order on $\{1, \ldots, n\}$ (where $n \in \mathbb{N}^*$) is denoted by O_n. An *almost transitive tournament* is a tournament obtained from a transitive tournament with at least three vertices by reversing the arc formed by its two extremal vertices.

For every finite sets E and F, we denote $E \subset F$ when E is a subset of F and $\mid E \mid$ the cardinality of E.

Given a tournament $T = (V, A)$, for each subset X of V we associate the *subtournament* of T induced by X, that is the tournament $T[X] = (X, A \cap (X \times X))$. For convenience, the subtournament $T[V \backslash X]$ is denoted by $T - X$, and by $T - x$ whenever $X = \{x\}$.

Let $T = (V, A)$ be a tournament, a subset I of V is an *interval* of T if for every $x \in V \setminus I$, x dominates or is dominated by all elements of I. For instance, $\emptyset$, V and $\{x\}$, where $x \in V$, are intervals of T called *trivial* intervals. A tournament is *indecomposable* if all its intervals are trivial; otherwise, it is *decomposable*. For example, the tournament $C_3 = (\{1, 2, 3\}, \{(1, 2), (2, 3), (3, 1)\})$ is indecomposable, whereas, the tournaments $C_4 = (\{1, 2, 3, 4\}, \{(1, 2), (2, 3), (3, 4), (4, 1), (3, 1), (2, 4)\})$, $\delta^+ = (\{1, 2, 3, 4\}, \{(1, 2), (2, 3), (3, 1), (1, 4), (2, 4), (3, 4)\})$ and $\delta^- = (\delta^+)^*$ are decomposable.

Given two tournaments $T = (V, A)$ and $T' = (V', A')$, an *isomorphism* from T onto T' is a bijection f from V onto V' satisfying: for any $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. The tournaments T and T' are *isomorphic* if there exists an isomorphism from one onto the other. This is denoted by $T \sim T'$. A tournament T' *embeds* into a tournament T (or T *embeds* T'), if T' is isomorphic to a subtournament of T. A 3-*cycle* (resp. 4-*cycle*) is a tournament which is isomorphic to C_3 (resp. C_4). Moreover, a *positive diamond* (resp. *negative diamond*) is a tournament that is isomorphic to δ^+ (resp. δ^-). A *diamond* is a positive or a negative diamond. For convenience, a set X of vertices of a tournament T is called diamond of T if $T[X]$ is a diamond.

1.2 Self duality and reconstruction

A tournament T on a set V is *self dual* if T and T^* are isomorphic, it's *strongly self dual* if for every subset X of V, $T[X]$ and $T^*[X]$ are isomorphic. For each non negative integer k, the tournament T is $(\leq k)$-*self dual* whenever for every set X of at most k vertices, the subtournament $T[X]$ is self dual. It is easy to see that a transitive tournament or an almost transitive tournament is strongly self dual. Conversely, Reid and Thomassen [23] was proved that a strongly self dual tournament with at least 8 vertices is transitive or almost transitive. This result was used by K. B. Reid and C. Thomassen [23] in order to characterize the pair of *hereditarily isomorphic* tournaments, that is, the pair of tournaments T , T' on a set V such that for every subset X of V, the subtournaments $T[X]$ and $T'[X]$ are isomorphic. A relaxed version of this notion is the following. Consider two tournaments T and T' on the same vertex set V, with $\mid V \mid = n \geq 2$. Let k be an non negative integer k with $k \leq n$. The tournaments T and T' are $\{k\}$-*hypomorphic*, whenever for every set X of k vertices, the subtournaments $T[X]$ and $T'[X]$ are isomorphic. If T and T' are $\{n-k\}$-hypomorphic, we say that T and T' are $\{-k\}$-*hypomorphic*. Let F be a set of integers. The tournaments T and T' are F-*hypomorphic*, if for every $p \in F$, T and T' are $\{p\}$-hypomorphic, in particular, if $F = \{0, \dots, k\}$, we say that T and T' are $(\leq k)$-*hypomorphic*. For example, every two tournaments on the same vertex set with at least 2 elements are (≤ 2)-hypomorphic. The tournament T is F-*reconstructible* provided that every tournament F-hypomorphic to T is isomorphic to T. This notion was introduced by R. Fraïssé [11] in 1970.

In 1972, G. Lopez ([15],[16]) showed that a tournament, with at least 6 vertices, is (≤ 6)-reconstructible (see also [17]). It follows from a "Combinatorial Lemma" of M. Pouzet (see Section 2) that a tournament, with at least 12 vertices, is $\{-6\}$-reconstructible.

On the other hand, P. K. Stockmeyer [25] showed that the tournaments are not, in general, $\{-1\}$-reconstructible, invalidating so the conjecture of Ulam [26] for tournaments. Then, M. Pouzet ([1],[2]) proposed the $\{-k\}$-reconstruction problem of tournaments. P. Ille [14] established that a tournament with at least 11 vertices is $\{-5\}$-reconstructible. G. Lopez and C. Rauzy ([18],[19]) showed that a tournament with at least 10 vertices is $\{-4\}$-reconstructible. The $\{-k\}$-reconstruction problem of tournaments is still open for $k \in \{2, 3\}$.

In 1995, Y. Boudabbous and A. Boussaïri [6] studied the $\{-3\}$ -reconstruction of decomposable tournaments, for which they give a partial positive answer.

In this paper, we find and improve, in a less complicated way, the results of this study. In order to present our results, we need the Gallai's decomposition theorem [12] .

1.3 Gallai's decomposition

Given a tournament $T = (V, A)$, we define on V a binary relation $\mathcal{R}$ as follows: for all $x \in V$, $x\mathcal{R}x$ and for $x \neq y \in V$, $x\mathcal{R}y$ if there exist two integers n, $m \geq 1$ and two sequences $x_0 = x, \ldots, x_n = y$ and $y_0 = y, \ldots, y_m = x$ of vertices of T such that $x_i \longrightarrow x_{i+1}$ for $i = 0, \ldots, n-1$ and $y_j \longrightarrow y_{j+1}$ for $j = 0, \ldots, m-1$. Clearly, $\mathcal{R}$ is an equivalence relation on V. The equivalence classes of $\mathcal{R}$ are called the *strongly connected components* of T. A tournament is then *strongly connected* if it has at most one strongly connected component, otherwise, it is *non-strongly connected*.

The next result is due to J. W. Moon [20].

Lemma 1 *[20] Given a strongly connected tournament $T = (V, A)$ with $n \geq 3$ vertices, for every integer $k \in \{3, \ldots, n\}$ and for every $x \in V$, there exists a subset X of V such that $x \in X$, $\mid X \mid= k$ and the subtournament $T[X]$ is strongly connected.*

Let T be a tournament on a set V. A partition $\mathcal{P}$ of V is an *interval partition* of T if all the elements of $\mathcal{P}$ are intervals of T. It ensues that the elements of $\mathcal{P}$ may be considered as the vertices of a new tournament, the *quotient* $T/\mathcal{P} = (\mathcal{P}, A/\mathcal{P})$ *of* T *by* $\mathcal{P}$, defined in the following way: for any $X \neq Y \in \mathcal{P}$, $(X, Y) \in A/\mathcal{P}$ if $(x, y) \in A$, for $x \in X$ and $y \in Y$. On another hand, a subset X of V is a *strong interval* of T provided that X is an interval of T and for every interval Y of T, if $X \cap Y \neq \emptyset$, then $X \subset Y$ or $Y \subset X$. Here, for each tournament $T = (V, A)$ with $\mid V \mid\geq 2$, $\mathcal{P}(T)$ denotes the family of maximal, strong intervals of T, under the inclusion, amongst the strong intervals of T distinct from V. Clearly, $\mathcal{P}(T)$ realizes an interval partition of T.

Consider a tournament $H = (V, A)$. For every $x \in V$ is associated the tournament $T_x = (V_x, A_x)$ such that the V_x's are mutually disjoint. The lexicographical sum of T_x's over H is the tournament T denoted by $H(T_x; x \in V)$ and defined on the union of V_x's as follows: given $u \in V_x$ and $v \in V_y$, where x, $y \in V$, (u, v) is an arc of T if either $x = y$ and $(u, v) \in A_x$ or $x \neq y$ and $(x, y) \in A$. This operation consists in fact to replace every vertex x of V by T_x so that V_x becomes an interval; we say that the vertex x is *dilated* by T_x. For example, an almost transitive tournament is obtained from a 3-cycle by dilating one of its vertices by a transitive tournament.

The Gallai's decomposition theorem [12] consists in the following examination of the quotient $T/\mathcal{P}(T)$.

Theorem 2 *([9],[12]) Let T be a tournament with at least two vertices.*

1. *The tournament T is non-strongly connected if and only if $T/\mathcal{P}(T)$ is transitive. In addition, if T is non-strongly connected, then $\mathcal{P}(T)$ is the family of the strongly connected components of T.*
2. *The tournament T is strongly connected if and only if $T/\mathcal{P}(T)$ is indecomposable and $\mid \mathcal{P}(T) \mid \geq 3$.*

We complete this subsection by the following notation.

Notation 3 *For every tournament T defined on a vertex set V with at least two elements, we associate the partition $\widetilde{\mathcal{P}}(T)$ of V defined from $\mathcal{P}(T)$ as follows:*

- *If T is strongly connected, $\widetilde{\mathcal{P}}(T) = \mathcal{P}(T)$.*
- *If T is non-strongly connected, a subset A of V belongs to $\widetilde{\mathcal{P}}(T)$ if and only if either $A \in \mathcal{P}(T)$ and $\mid A \mid \geq 2$, or A is a maximal union of consecutive vertices of the transitive tournament $T/\mathcal{P}(T)$ which are singletons.*

1.4 Statement of the results

In this paper, we begin by proving the following theorem (Section 3). That improves the result obtained by K. B. Reid and C. Thomassen [23]. (Note that, the study of [23] is easly reduced to the decomposable case).

Theorem 4 *A decomposable tournament which has at least 9 vertices is $\{-3\}$-self dual if and only if it is either transitive or almost transitive.*

Then, we characterize each tournament $\{-3\}$-hypomorphic to a decomposable tournament with at least 9 vertices (Section 4). For the statement, we need additional notation.

Notation 5 *Consider a set P of non zero integers, an integer $n \geq 6$ and $q \in \{2, 3\}$. We denote by:*

- *$I_{n,P}$, the class of tournaments with n vertices which are indecomposable, not self dual and $\{p\}$-self dual for every $p \in P$; these tournaments being considered up to an isomorphism.*

- $C_3(I_{n,P})$ *(resp.* $O_q(I_{n,P})$*), the class of tournaments with* $n+2$ *(resp.* $n+q-1$*) vertices obtained, from the* 3*-cycle* C_3 *(resp. from the transitive tournament* O_q*), by dilating one of its vertices by a tournament belonging to the class* $I_{n,P}$*; these tournaments being considered up to an isomorphism.*
- *For each integer* $m \geq 8$*, we denote by* Ω_m *the union* $C_3(I_{m-2,\{-1,-2,-3\}}) \cup O_3(I_{m-2,\{-1,-2,-3\}}) \cup O_2(I_{m-1,\{-2,-3\}})$.

Here is the characterization.

Theorem 6 *Consider a decomposable tournament* T *with* $n \geq 9$ *vertices and let* T' *be a tournament* $\{-3\}$*-hypomorphic to* T*. Then, we have:*

1. $\widetilde{\mathcal{P}}(T') = \widetilde{\mathcal{P}}(T)$ *and one of the following situations is achieved.*
   (a) T *is almost transitive and* $T' \sim T$.
   (b) T *is not almost transitive,* $T/\widetilde{\mathcal{P}}(T) = T'/\widetilde{\mathcal{P}}(T)$ *and one of the following situations is achieved.*
      (i) $T \notin \Omega_n$*, for every* $X \in \widetilde{\mathcal{P}}(T)$*,* $T'[X] \sim T[X]$ *and* $T' \sim T$.
      (ii) $T \in \Omega_n$ *and* $T' \not\sim T$.
2. T *is not* $\{-3\}$*-reconstructible if and only if* $T \in \Omega_n$.

We deduce the following results.

Corollary 7 *Let* T *be a decomposable tournament on a set* V*, with* $\mid V \mid \geq 9$*. If there is no interval* X *of* T *such that* $T[X]$ *is indecomposable and* $\mid V \setminus X \mid \leq 2$*, then* T *is* $\{-3\}$*-reconstructible.*

Corollary 8 *If there exists an integer* $n_0 \geq 7$ *for which the indecomposable tournaments with at least* n_0 *vertices are* $\{-3\}$*-reconstructible, then the tournaments with at least* $n_0 + 2$ *vertices are* $\{-3\}$*-reconstructible.*

By Theorem 17 (Section 2) and Corollary 8, we reduce the $\{-3\}$-reconstruction problem of tournaments to the indecomposable case between a tournament and its dual.

2 Preliminary results

In this section, we recall and prove some results which will be used in next sections.

First, concerning the decomposability, we recall the following notation, lemma and corollary.

Notation 9 *Given a tournament $T = (V, A)$, for each subset X of V, such that $| X |\geq 3$ and $T[X]$ is indecomposable, we associate the following subsets of $V \setminus X$.*

- *$Ext(X)$ is the set of $x \in V \setminus X$ such that $T[X \cup \{x\}]$ is indecomposable.*
- *$[X]$ is the set of $x \in V \setminus X$ such that X is an interval of $T[X \cup \{x\}]$.*
- *For every $u \in X$, $X(u)$ is the set of $x \in V \setminus X$ such that $\{u, x\}$ is an interval of $T[X \cup \{x\}]$.*

Lemma 10 *[10] Let $T = (V, A)$ be a tournament and let X be a subset of V such that $| X |\geq 3$ and $T[X]$ is indecomposable. The family $Ext(X), [X]$ and $X(u)$ where $u \in X$ constitutes a partition of $V \setminus X$. (Some elements of this family can be empty).*

The next result follows from Lemma 10.

Corollary 11 *[10] Let $T = (V, A)$ be an indecomposable tournament. If X is a subset of V, such that $| X |\geq 3$, $| V \setminus X |\geq 2$ and $T[X]$ is indecomposable, then there are distinct $x, y \in V \setminus X$ such that $T[X \cup \{x, y\}]$ is indecomposable.*

Second, we consider the following remark and notation.

Remark 12 **i)** *Up to an isomorphism, there exist four tournaments with four vertices: O_4, C_4, δ^+ and δ^-. In addition, both of them is decomposable.*

ii) *Given two $\{3\}$-hypomorphic tournaments T and T' with the same vertex set V with $| V |\geq 4$, T and T' are (≤ 4)-hypomorphic if and only if for every subset X of V, if $T[X]$ or $T'[X]$ is a diamond, then $T'[X] \sim T[X]$.*

iii) *Consider two $\{3\}$-hypomorphic tournaments T and T'. If T is without diamonds then T' and T are $\{4\}$-hypomorphic.*

Notation 13 *Given a tournament $T = (V, A)$, a subset F of V and a tournament H, we denote by $S(T, H; F) = \{X \subset V;\, F \subset X$ and $T[X] \sim H\}$ and $n(T, H; F) =| S(T, H; F) |$.*

Then, we recall the following "Combinatorial Lemma" of M. Pouzet [21].

Lemma 14 *[21] Let p, r be two positive integers, E be a set of at least $p + r$ elements and U, U' be two sets of subsets of p elements of E. If for each subset Q of E with $| Q |= p+r$, the number of the elements of U which* *are contained in Q is equal to the number of the elements of U' which are contained in Q, then for every finite subsets P' and Q' of E, such that P' is contained in Q' and $Q' \setminus P'$ has at least $p + r$ elements, the number of elements of U containing P' and included in Q' is equal to the number of elements of U' containing P' and included in Q'. In particular, if E has at least $2p + r$ elements, then U and U' are equal.*

From the Combinatorial Lemma, we have the next corollaries.

Corollary 15 *[22] Consider positive integers n, p, h such that $p < n$ and $h \leq n - p$, a tournament H with h vertices and two tournaments T and T' defined on the same vertex set V with $\mid V \mid = n$. If T and T' are $\{-p\}$-hypomorphic then for each subset X of V of at most p elements, $n(T', H; X) = n(T, H; X)$.*

Corollary 16 *[21] Consider two tournaments $T = (V, A)$ and $T' = (V, A')$ and an integer p such that $0 < p <\mid V \mid$. If T and T' are $\{p\}$-hypomorphic, then T and T' are $\{q\}$-hypomorphic for each $q \in \{1, \ldots, min(p, \mid V \mid -p)\}$. In particular, if $\mid V \mid \geq 6$ and T and T' are $\{-3\}$-hypomorphic, then T and T' are (≤ 3)-hypomorphic.*

Now, recall the following theorem, called the "Inversion Theorem", which was obtained by A. Boussaïri, P. Ille, G. Lopez and S. Thomassé ([8],[9]).

Theorem 17 *([8],[9]) Given an indecomposable tournament T with at least 3 vertices, the only tournaments which are $\{3\}$-hypomorphic to T are T and T^*.*

The following corollary is a consequence of Theorem 17.

Corollary 18 *[9] Let T and T' be two $\{3\}$-hypomorphic tournaments with at least 3 verices.*

i) *$\mathcal{P}(T) = \mathcal{P}(T')$.*

ii) *T is strongly connected (resp. indecomposable) if and only if T' is strongly connected (resp. indecomposable).*

iii) *If T is strongly connected, then the quotients $T'/\mathcal{P}(T)$ and $T/\mathcal{P}(T)$ are either equal or dual.*

From this corollary, we obtain the following remark.

Remark 19 *Let T and T' be two $\{3\}$-hypomorphic tournaments on a set V with $\mid V \mid \geq 3$, and I be a subset of V such that $T[I]$ is strongly connected. If I is an interval of T, then I is an interval of T'.*

Given a tournament T on a set V and a subset I of V, we denote by I_T^+ (resp. I_T^-) the set of vertices $x \in V \setminus I$ such that $I \longrightarrow x$ (resp. $x \longrightarrow I$).

We complete this section by the following result.

Proposition 20 *Let T and T' be two tournaments on a set V with $|V| \geq 6$, and I be an interval of T such that $\mid I \mid \geq 3$ and $T[I]$ is indecomposable.*

i) *If T and T' are $\{3,-2\}$-hypomorphic (resp. $\{-3\}$-hypomorphic) and $|V \setminus I| \geq 2$ (resp. $|V \setminus I| \geq 3$), then $T[I] \backsim T'[I]$.*

ii) *If T and T' are $\{3,-2\}$-hypomorphic (resp. $\{-3\}$-hypomorphic) and $|V \setminus I| \geq 3$ (resp. $|V \setminus I| \geq 4$), then $\left|I_T^+\right| = \left|I_{T'}^+\right|$ and $\left|I_T^-\right| = \left|I_{T'}^-\right|$.*

iii) *If T and T' are $\{-3\}$-hypomorphic and $|V \setminus I| \geq 4$, then $T[I] \backsim T'[I]$, $I_T^+ = I_{T'}^+$ and $I_T^- = I_{T'}^-$.*

Proof.

First, note that if T and T' are $\{-3\}$-hypomorphic, then T and T' are $\{3\}$-hypomorphic (by Corollary 16). Moreover, as $T[I]$ is indecomposable (in particular, it is strongly connected because $\mid I \mid \geq 3$) and $T[I]$ and $T'[I]$ are $\{3\}$-hypomorphic, then by Corollary 18, $T'[I]$ is indecomposable and by Remark 19, I is an interval of T'.

i) Let $a \neq b \in I$ and J be a subset of V containing $\{a,b\}$ such that $T[J]$ is indecomposable and $\mid I \mid = \mid J \mid$ and denote by H the subtournament $T[I]$. As I is an interval of T, then $I \cap J$ is an interval of $T[J]$. However, $\{a,b\} \subset I \cap J$ and $T[J]$ is indecomposable, then $I \cap J = J$ and hence $I = J$ (because $\mid I \mid = \mid J \mid$). Thus, I is the only subset J of V containing $\{a,b\}$ such that $T[J]$ is indecomposable and $\mid J \mid = \mid I \mid$. In particular $S(T,H;\{a,b\}) = \{I\}$ and then $n(T,H;\{a,b\}) = 1$.
By interchanging T and T' in the previous result, I is the only subset J of V containing $\{a,b\}$ such that $T'[J]$ is indecomposable and $\mid I \mid = \mid J \mid$. In particular, $S(T',H;\{a,b\}) \subset \{I\}$. Lastly, as T and T' are $\{-2\}$-hypomorphic (resp. $\{-3\}$-hypomorphic), $\mid V \mid \geq 6$ and $\mid I \mid \leq \mid V \mid -2$ (resp. $\mid I \mid \leq \mid V \mid -3$), then by Corollary 15, $n(T,H;\{a,b\}) = n(T',H;\{a,b\})$. As $n(T,H;\{a,b\}) = 1$, then $S(T',H;\{a,b\}) \neq \emptyset$ and so $S(T',H;\{a,b\}) = \{I\}$. Consequently, $T'[I] \sim T[I]$.

ii) Let $a \neq b \in I$ and denote by H a tournament with vertex set $I \cup \{u\}$ (where $u \notin I$) such that $H[I] = T[I]$ and $I \longrightarrow u$. Clearly, if $I_T^+ \neq \emptyset$, then for each $x \in I_T^+$, $I \cup \{x\} \in S(T,H;\{a,b\})$. Conversely, assume that $S(T,H;\{a,b\}) \neq \emptyset$ and consider an element J of $S(T,H;\{a,b\})$. Let f be an isomorphism from H to $T[J]$ and let $\alpha = f(u)$. As I is the unique non trivial interval of H and $f(I) = J \setminus \{\alpha\}$, then $J \setminus \{\alpha\}$ is the unique non trivial interval of $T[J]$. However, $I \cap J$ is an interval of $T[J]$ and $\{a,b\} \subset I \cap J$, then $I \cap J = J \setminus \{\alpha\}$ and hence $J \setminus \{\alpha\} \subset I$. So, $J \setminus \{\alpha\} = I$ (because $\mid I \mid = \mid J \setminus \{\alpha\} \mid$). Thus, $J = I \cup \{\alpha\}$ and $\alpha \in I_T^+$.

We conclude that $S(T,H;\{a,b\}) = \{I \cup \{x\};\, x \in I_T^+\}$ and hence, $n(T,H;\{a,b\}) = \mid I_T^+ \mid$.

As $T'[I] \sim T[I]$ (by i)), then by interchanging T and T' in the previous result, we deduce that $n(T',H;\{a,b\}) = \mid I_{T'}^+ \mid$. Lastly, as T and T' are $\{-2\}$-hypomorphic (resp. $\{-3\}$-hypomorphic) and $\mid I \cup \{u\} \mid \leq \mid V \mid -2$ (resp. $\mid I \cup \{u\} \mid \leq \mid V \mid -3$), then by Corollary 15, $n(T,H;\{a,b\}) = n(T',H;\{a,b\})$. Therefore, $\mid I_T^+ \mid = \mid I_{T'}^+ \mid$ and hence, $\mid I_T^- \mid = \mid I_{T'}^- \mid$.

iii) By ii), we have $\left|I_T^+\right| = \left|I_{T'}^+\right|$. Assume now that $I_T^+ \setminus I_{T'}^+ \neq \emptyset$ and let $x \in I_T^+ \setminus I_{T'}^+$. As I is an interval of T', then $x \in I_{T'}^-$. The tournaments $T-x$ and $T'-x$ are $\{3,-2\}$-hypomorphic, $\mid V \setminus \{x\} \mid \geq 6$, and $|(V \setminus \{x\}) \setminus I| \geq 3$, then by ii), $\left|I_{T-x}^+\right| = \left|I_{T'-x}^+\right|$. However, $\left|I_{T-x}^+\right| = \left|I_T^+\right| - 1$ and $\left|I_{T'-x}^+\right| = \left|I_{T'}^+\right|$; contradiction. It follows that $I_T^+ \subset I_{T'}^+$ and then $I_T^+ = I_{T'}^+$. By duality, we obtain $I_T^- = I_{T'}^-$.

□

3 Proof of Theorem 4

For the tournaments without diamonds, H. Bouchaala and Y. Boudabbous [5] established the following result.

Proposition 21 *[5] Given a tournament T without diamonds and which has at least 9 vertices, T is $\{-3\}$-self dual if and only if it is strongly self dual.*

We present now some results concerning tournaments embedding a diamond.

Remark 22 *A diamond δ has a unique non trivial interval I. Moreover, $\delta[I]$ is a 3-cycle.*

Lemma 23 *[4] If $T = (V, A)$ is a tournament embedding a diamond, then each vertex of T is contained in at least one diamond of T.*

The following proposition was obtained by M. Sghiar in 2004. This result plays an important role in the proof of Theorem 4.

Proposition 24 *[24] Let T be a tournament, with at least 8 vertices, embedding a diamond. If T has an interval of cardinality 2, then T is not $\{-3\}$-self dual.*

For the proof of this proposition, we need some definitions and notations. Given a tournament $T = (V, A)$, if $X = \{a, b, c, d\}$ is a subset of V such that $T[X]$ is a diamond and $T[\{a, b, c\}]$ is a 3-cycle, we say that X is a diamond of T of *center* d and cycle $\{a, b, c\}$. Let $x \neq y \in V$, we denote:

- $\delta^+_{T,\{x,y\}}$ (resp. $\delta^-_{T,\{x,y\}}$), the number of positive (resp. negative) diamonds of T whose cycle contains $\{x, y\}$.
- $C_{T,\{x,y\}}$, the set of elements w of V such that $T[\{x, y, w\}]$ is a 3-cycle.
- $\delta^+_{T,\{x,y,w\}}$ (resp. $\delta^-_{T,\{x,y,w\}}$), the number of positive (resp. negative) diamonds of T whose cycle is $\{x, y, w\}$, where w is an element of $C_{T,\{x,y\}}$.
- $D^+_{T,\{x\}}(y)$ (resp. $D^-_{T,\{x\}}(y)$), the number of positive (resp. negative) diamonds of T passing by x and whose center is y.
- $D^+_{T,\{x,y\}}$ (resp. $D^-_{T,\{x,y\}}$), the number of positive (resp. negative) diamonds of T passing by x and y.
- $\delta^+_T(x)$ (resp. $\delta^-_T(x)$), the number of positive (resp. negative) diamonds of T whose center is x.

Lemma 25 *[24] Let $T = (V, A)$ be a $\{-3\}$-self dual tournament with at least 7 vertices. If T embeds a diamond, then every vertex of T is the center of at least one diamond of T.*

Proof. Suppose for a contradiction that there exists a vertex x of T such that $\delta^+_T(x) = \delta^-_T(x) = 0$. From Lemma 23, there exists a diamond σ of T containing x. By interchanging T and T^*, we can assume that σ is a negative diamond. Let y be the center of σ. So, $D^-_{T,\{x\}}(y) \neq 0$ and $D^+_{T,\{y\}}(x) = 0$. If $C_{T,\{x,y\}} = \emptyset$, then $\delta^+_{T,\{x,y\}} = \delta^-_{T,\{x,y\}} = 0$. If $C_{T,\{x,y\}} \neq \emptyset$, then pick $w \in C_{T,\{x,y\}}$ and let $X = \{x, y, w\}$. As T and T^* are $\{-3\}$-hypomorphic and $\mid V \mid \geq 7$, then from Corollary 15, $n(T, \delta^+; X) = n(T^*, \delta^+; X)$. So, $n(T, \delta^+; X) = n(T, \delta^-; X)$ and hence $\delta^+_{T,X} = \delta^-_{T,X}$. However, $\delta^+_{T,\{x,y\}} = \sum\limits_{w \in C_{T,\{x,y\}}} \delta^+_{T,\{x,y,w\}}$ and $\delta^-_{T,\{x,y\}} = \sum\limits_{w \in C_{T,\{x,y\}}} \delta^-_{T,\{x,y,w\}}$. Thus, $\delta^+_{T,\{x,y\}} = \delta^-_{T,\{x,y\}}$. On the other hand, we have $D^-_{T,\{x,y\}} = n(T, \delta^-; \{x, y\}) = n(T^*, \delta^+; \{x, y\})$, $D^+_{T,\{x,y\}} = n(T, \delta^+; \{x, y\})$ and from Corollary 15, $n(T, \delta^+; \{x, y\}) = n(T^*, \delta^+; \{x, y\})$, hence $D^+_{T,\{x,y\}} = D^-_{T,\{x,y\}}$. However, $D^+_{T,\{x,y\}} = D^+_{T,\{y\}}(x) + \delta^+_{T,\{x,y\}}$, $D^-_{T,\{x,y\}} = D^-_{T,\{x\}}(y) + \delta^-_{T,\{x,y\}}$ and $\delta^+_{T,\{x,y\}} = \delta^-_{T,\{x,y\}}$, thus, $D^+_{T,\{y\}}(x) = D^-_{T,\{x\}}(y)$; which contradicts the fact that $D^-_{T,\{x\}}(y) \neq 0$ and $D^+_{T,\{y\}}(x) = 0$. □

Lemma 26 *[24] Consider a $\{-2\}$-self dual (resp. $\{-3\}$-self dual) tournament $T = (V, A)$ with at least 7 (resp. 8) vertices and two distinct vertices a, b of T. If $\{a, b\}$ is an interval of T then $\delta^+_T(a) = \delta^-_T(a)$.*

Proof. Let H be the tournament obtained from one positive diamond by dilating its center by a tournament with 2 vertices. Let $\Delta^+_T(a) = \{X \subset V; T[X]$ is a positive diamond with center $a\}$. Let X be an element of $\Delta^+_T(a)$. As $\{a, b\}$ is an interval of T, so $\{a, b\} \cap X$ is an interval of $T[X]$. Then, by Remark 22, $b \notin X$ and $\{a, b\}$ is an interval of $T[X \cup \{b\}]$. Hence, $X \cup \{b\} \in S(T, H; \{a, b\})$. Let's consider the map $f : \Delta^+_T(a) \longrightarrow S(T, H; \{a, b\})$ defined by: for each $X \in \Delta^+_T(a)$, $f(X) = X \cup \{b\}$. Clearly, f is bijective and so $\delta^+_T(a) = n(T, H; \{a, b\})$. By interchanging T and T^*, we deduce that $\delta^-_T(a) = \delta^+_{T^*}(a) = n(T^*, H; \{a, b\})$. On the other hand, as T and T^* are $\{-2\}$-hypomorphic (resp. $\{-3\}$-hypomorphic) and $\mid V \mid \geq 7$ (resp. $\mid V \mid \geq 8$), then from Corollary 15, $n(T, H; \{a, b\}) = n(T^*, H; \{a, b\})$. Thus, $\delta^+_T(a) = \delta^-_T(a)$. □

Proof of Proposition 24. Assume by contradiction, that T is $\{-3\}$-self dual and has an interval $\{a, b\}$ with $a \neq b$. By Lemma 25, T has a diamond $T[X]$ with center a. Assume for example that $T[X]$ is a positive diamond. Clearly, $b \notin X$. Consider a vertex $x \in X \setminus \{a\}$. As $T - x$ (resp. T) is $\{-2\}$-self dual (resp. $\{-3\}$-self dual) and $\{a, b\}$ is an interval of $T - x$ (resp. T), then, by Lemma 26, $\delta^+_{T-x}(a) = \delta^-_{T-x}(a)$ (resp. $\delta^+_T(a) = \delta^-_T(a)$). So, $0 = \delta^+_T(a) - \delta^-_T(a) = \delta^+_{T-x}(a) + D^+_{T,\{x\}}(a) - \delta^-_{T-x}(a) = D^+_{T,\{x\}}(a)$; contradiction. □

Theorem 4 is an immediate consequence of Proposition 21 and the below proposition.

Proposition 27 *Every decomposable tournament with at least 8 vertices embedding a diamond is not $\{-3\}$-self dual.*

Proof. Let $T = (V, A)$ be a decomposable tournament with at least 8 vertices embedding a diamond. Assume by contradiction that T is $\{-3\}$-self dual. By Proposition 24, T has no interval of size 2. Let I be a minimal (under the inclusion) non trivial interval of T. Clearly, the subtournament $T[I]$ is indecomposable. And then it is strongly connected. Firstly, assume that $|V \setminus I| \geq 4$. Let $z \in V \setminus I$ and suppose, for example that $z \longrightarrow I$ in T. As T and T^* are $\{-3\}$-hypomorphic, then from Proposition 20, $z \longrightarrow I$ in T^*; contradiction. Secondly, assume that $|V \setminus I| \leq 3$. Let $k = \mid V \setminus I \mid$. We have so $k \in \{1, 2, 3\}$ and $\mid I \mid \geq 8 - k$. As the subtournament $T[I]$ is strongly connected and $8 - k \in \{5, 6, 7\}$, then from Lemma 1, there exists a subset

X of I such that $\mid X \mid= 4-k$ and $T[I]-X$ is strongly connected. Let Y be a subset of $V \setminus I$ such that $\mid Y \mid= k-1$. Clearly, the subtournament $T-(X \cup Y)$ is not self dual; contradicts the fact that $\mid X \cup Y \mid= 3$. □

4 Proof of Theorem 6

The proof of Theorem 6 is based on the next result.

Proposition 28 *Let T be a strongly connected and decomposable tournament on a set V with $\mid V \mid= n \geq 9$, which is not almost transitive. If T' is a tournament $\{-3\}$-hypomorphic to T, then the following assertions hold.*

1. *$\mathcal{P}(T') = \mathcal{P}(T)$ and $T'/\mathcal{P}(T) = T/\mathcal{P}(T)$.*
2. *If there exists $X \in \mathcal{P}(T)$ such that $T'[X] \not\sim T[X]$, then $\mid \mathcal{P}(T) \mid= 3$ and $\mid X \mid= n-2$.*
3. *If for each $X \in \mathcal{P}(T)$, $\mid X \mid\leq n-3$, then for each $X \in \mathcal{P}(T)$, $T'[X] \sim T[X]$ and in particular $T' \sim T$.*

For the proof of the last proposition, we use the following remark and lemma.

Remark 29 *Let T and T' be two tournaments on a set V with $\mid V \mid\geq 3$ and Γ be a common interval partition of T and T' such that $T/\Gamma = T'/\Gamma$. Given a non empty subset A of V and let $\Gamma_A = \{X \cap A;\ X \in \Gamma$ and $X \cap A \neq \emptyset\}$. Then Γ_A is a common interval partition of $T[A]$ and $T'[A]$ and $T[A]/\Gamma_A = T'[A]/\Gamma_A$. Suppose that for each $Y \in \Gamma_A$, there exists an isomorphism φ_Y from $T[Y]$ onto $T'[Y]$ and consider the map $f : A \longrightarrow A$ defined by: for each $x \in A$, $f(x) = \varphi_Y(x)$ where Y is the unique element of Γ_A such that $x \in Y$. Then, f is an isomorphism from $T[A]$ onto $T'[A]$. In particular, if for each $X \in \Gamma$, $T[X]$ and $T'[X]$ are hereditarily isomorphic, then T and T' are hereditarily isomorphic.*

Lemma 30 *[3] Let $T = (V, A)$ and $T' = (V', A')$ be two isomorphic tournaments, f be an isomorphism from T onto T', $i \in V$ and R_i (resp. R'_i) be a tournament defined on a vertex set I_i (resp. I'_i) disjoint from V (resp. V'). Let R (resp. R') be the tournament obtained from T (resp. T') by dilating the vertex i (resp. $f(i)$) by R_i (resp. R'_i). Then $R \sim R'$ if and only if $R_i \sim R'_i$.*

Note that this lemma is a simple generalization of a result communicated by A. Boussaïri, and on which $V' = V$ and $f = id_V$.

Proof of Proposition 28.

1. By Corollary 16, T and T' are (≤ 3)-hypomorphic. So, from Corollary 18, $\mathcal{P}(T) = \mathcal{P}(T')$ and $T'/\mathcal{P}(T) = T/\mathcal{P}(T)$ or $T'/\mathcal{P}(T) = T^*/\mathcal{P}(T)$. Assume by contradiction that $T'/\mathcal{P}(T) = T^*/\mathcal{P}(T)$. In this case, we are going to show that for every $X \in \mathcal{P}(T)$, $T[X]$ is transitive, and thus by Remark 29, T' is hereditarily isomorphic to T^*. Since T' is $\{-3\}$-hypomorphic to T, then T is $\{-3\}$-self dual. By Theorem 4, the tournament T is almost transitive; which contradicts the hypothesis. For that, proceed by contradiction and consider an element X of $\mathcal{P}(T)$ and a subset $\{\alpha, \beta, \gamma\}$ of X such that $T[\{\alpha, \beta, \gamma\}]$ is a 3-cycle. As T is strongly connected, there is $a \in V \setminus X$ such that $X \longrightarrow a$. From Corollary 15, $n(T, \delta^+; \{\alpha, \beta, a\}) = n(T', \delta^+; \{\alpha, \beta, a\})$. Moreover, $n(T, \delta^+; \{\alpha, \beta, a\}) \neq 0$, because $T[\{\alpha, \beta, \gamma, a\}] \sim \delta^+$, then there exists a subset K of V such that $\{\alpha, \beta, a\} \subset K$ and $T'[K] \sim \delta^+$. Hence, $\mid K \cap X \mid= 3$, because otherwise $K \cap X$ is an interval with two elements of $T'[K]$; which contradicts Remark 22. So, $T'[K]$ is written: $(K \cap X) \longleftarrow a$; which contradicts the fact that $T'[K] \sim \delta^+$.

2. By 1, we have $\mathcal{P}(T') = \mathcal{P}(T)$ and $T'/\mathcal{P}(T) = T/\mathcal{P}(T)$. We distinguish the following two cases.

   - If for every $X \in \mathcal{P}(T)$, $\mid X \mid \leq n - |\mathcal{P}(T)| - 2$.
     Let $X \in \mathcal{P}(T)$ and let H be the tournament obtained from $T/\mathcal{P}(T)$ by dilating the vertex X by $T[X]$. Assume that $\mid X \mid \geq 2$ and consider a subset A of X with 2 elements. Consider a subset B of V containing X such that: $\forall Y \in \mathcal{P}(T) \setminus \{X\}$, $\mid B \cap Y \mid= 1$. Clearly, $B \in S(T, H; A)$ and hence $n(T, H; A) \neq \emptyset$. From Corollary 15, $n(T', H; A) = n(T, H; A)$, then $n(T', H; A) \neq 0$. So, there exists a subset K of V such that: $A \subset K$ and $T'[K] \sim H$. Then $\mathcal{P}(T'[K])$ have a unique element J non reduced to a singleton. Clearly, $T'[J] \sim T[X]$, in particular, $\mid J \mid=\mid X \mid$.
     Let $P_1 = \{Y \in \mathcal{P}(T); \mid Y \cap K \mid \geq 2\}$ and $P_2 = \{Y \in \mathcal{P}(T);$ $\mid Y \cap K \mid= 1\}$. The set K is the union of the two disjoint sets $K_1 = \bigcup\limits_{Y \in P_1} K \cap Y$ and $K_2 = \bigcup\limits_{Y \in P_2} K \cap Y$. As $A \subset X \cap K$, then $X \in P_1$ and so, $P_2 \subset (\mathcal{P}(T) \setminus \{X\})$, in particular, $\mid P_2 \mid \leq \mid \mathcal{P}(T) \mid -1$. For all $Y \in P_1$, $Y \cap K$ is a non trivial interval of $T'[K]$, then $Y \cap K \subset J$, so $K_1 \subset J$ and thus $(K \setminus J) \subset K_2$.
     Thus, $\mid \mathcal{P}(T) \mid -1 =\mid K \setminus J \mid \leq \sum\limits_{Y \in P_2} \mid K \cap Y \mid=\mid P_2 \mid \leq \mid \mathcal{P}(T) \mid -1$.
     So, $\mid P_2 \mid=\mid \mathcal{P}(T) \mid -1$ and then $P_2 = \mathcal{P}(T) \setminus \{X\}$ and $P_1 = \{X\}$. Thus, $K = (X \cap K) \cup K_2$. So, $\mid X \cap K \mid=\mid K \mid - \mid K_2 \mid=$ $\mid K \mid - \mid P_2 \mid=\mid K \mid - \mid \mathcal{P}(T) \mid +1 =\mid J \mid$. As in addition

$X \cap K \subset J$, then, $X \cap K = J$. So, $J \subset X$ and then $J = X$ because $\mid J \mid = \mid X \mid$. Consequently, $T'[X] \sim T[X]$.

- If there exists an element X of $\mathcal{P}(T)$ such that $\mid X \mid > n - |\mathcal{P}(T)| - 2$.
  In this case, it is clear that for every $Y \in \mathcal{P}(T) \setminus \{X\}$, $\mid Y \mid \leq 3$, so $T'[Y]$ and $T[Y]$ are hereditarily isomorphic. Suppose that $\mid V \setminus X \mid \geq 3$. Let x be an element of X and B be a subset of $V \setminus X$ such that $\mid B \mid = 3$. Denote by $V_{(x,B)}$ the set $(V \setminus (X \cup B)) \cup \{x\}$ and by $T_{(x,B)}$ (resp. $T'_{(x,B)}$) the subtournament $T[V_{(x,B)}]$ (resp. $T'[V_{(x,B)}]$). The tournament $T - B$ (resp. $T' - B$) is obtained from the tournament $T_{(x,B)}$ (resp. $T'_{(x,B)}$) by dilating the vertex x by $T[X]$ (resp. $T'[X]$). Moreover, by Remark 29, there exists an isomorphism g from $T_{(x,B)}$ onto $T'_{(x,B)}$ such that $g(x) = x$. As in addition, $T - B$ and $T' - B$ are isomorphic, then, from Lemma 30, $T'[X]$ is isomorphic to $T[X]$; which permits to conclude.

3. Is a direct consequence of 2.

□

Lemma 31 *Every strongly connected and decomposable tournament, which has 8 vertices, is $\{-2, -3\}$-reconstructible.*

Proof. Let H be a strongly connected and decomposable tournament defined on a vertex set X with $\mid X \mid = 8$ and let H' be a tournament $\{-2, -3\}$-hypomorphic to H. The tournaments H and H' are $\{3\}$-hypomorphic, by Corollary 16, so they are $\{3, 5, 6\}$-hypomorphic. Besides, by Corollary 18, $\mathcal{P}(H) = \mathcal{P}(H')$, and $H'/\mathcal{P}(H) = H/\mathcal{P}(H)$ or $H'/\mathcal{P}(H) = H^*/\mathcal{P}(H)$. Let's put $Q = \mathcal{P}(H)$, and discuss according to its cardinal.

- If $\mid Q \mid > 3$.
  If for every $Z \in Q$, $\mid Z \mid \leq 3$. By (≤ 3)-hypomorphy, $H'[Z] \sim H[Z]$; $\forall Z \in Q$. If $H'/Q = H/Q$, clearly $H' \sim H$. Suppose hence that $H'/Q = H^*/Q$. In this case, H' is hereditarily isomorphic to H^*. Then, H is $\{-3\}$-self dual. From Proposition 27, H is without diamonds. By Remark 12, iii), H is $\{4\}$-self dual. Thus, H is $\{4, 5, 6\}$-self dual. So, H is (≤ 6)-self dual and it is thus self dual, by the (≤ 6)-reconstruction of tournaments with at least 6 vertices [17]. It follows that $H' \sim H$.
  If there exists $Y \in Q$ such that $\mid Y \mid \geq 4$. In this case, as $\mid X \mid = 8$, $\mid Q \mid > 3$ and every tournament with 4 vertices is decomposable, then $\mid Q \mid = 5$, $\mid Y \mid = 4$ and for every $Z \in Q \setminus \{Y\}$, $\mid Z \mid = 1$.

- If $H'/Q = H/Q$. For $z \in X \setminus Y$, as $\mid Y \cup \{z\} \mid = \mid X \mid -3$, then, $H[Y \cup \{z\}] \sim H'[Y \cup \{z\}]$. It follows that $H'[Y] \sim H[Y]$, and then $H' \sim H$.
- If $H'/Q = H^*/Q$.
    - If $H[Y]$ is not a diamond. In this case, clearly $H'[Y]$ is hereditarily isomorphic to $H^*[Y]$ and hence H' and H^* are hereditarily isomorphic. As H' is $\{-3\}$-hypomorphic to H, then H is $\{-3\}$-self dual. From Proposition 27, H is then without diamonds, and it is clearly self dual and thus, $H' \sim H$.
    - If $H[Y]$ is a diamond. In this case, if $H'[Y]$ is not isomorphic to $H[Y]$, then H' is hereditarily isomorphic to H^* and so, H is $\{-3\}$-self dual; which contradicts Proposition 27. So, $H'[Y] \sim H[Y]$. For $z \in X \setminus Y$, it is easy to verify that $H'[Y \cup \{z\}]$ is not isomorphic to $H[Y \cup \{z\}]$; which contradicts the $\{-3\}$-hypomorphy between H' and H.

- If $\mid Q \mid = 3$. We distinguish the following two sub-cases.
    - If there exists an unique $Y \in Q$ such that $\mid Y \mid > 1$.
      In this case, Q is written: $Q = \{\{a\}, \{b\}, Y\}$ where $\mid Y \mid = 6$ and $a \longrightarrow Y \longrightarrow b$ in H. As $\mid Y \mid = 6$, we have: $H'[Y] \sim H[Y]$ by $\{-2\}$-hypomorphy. Thus, $H' \sim H$.
    - If there are $Y \neq Z \in Q$ such that $\min(\mid Y \mid, \mid Z \mid) > 1$.
      In this case, we can write: $Q = \{Y_1, Y_2, Y_3\}$ with: $\mid Y_3 \mid \leq \mid Y_2 \mid \leq \mid Y_1 \mid$, $\mid Y_2 \mid \geq 2$ and $\mid Y_1 \mid \geq 3$. By considering a subset A_1 (resp. A_2) with 3 (resp. 2) elements of Y_1 (resp. Y_2) and an element y_3 of Y_3, we see that the isomorphy between $H[A_1 \cup A_2 \cup \{y_3\}]$ and $H'[A_1 \cup A_2 \cup \{y_3\}]$ requires that: $H'/Q = H/Q$. So, if $\mid Y_1 \mid = 3$, then for every $i \in \{1, 2, 3\}$, $H'[Y_i] \sim H[Y_i]$ and then $H' \sim H$. Suppose thus that $\mid Y_1 \mid \geq 4$. As $\mid X \mid = 8$, then $\mid Y_1 \mid \in \{4, 5\}$ and $\mid Y_2 \mid \leq 3$. By considering an element y_2 of Y_2, we see that the isomorphy between $H[Y_1 \cup \{y_2\}]$ and $H'[Y_1 \cup \{y_2\}]$ requires the isomorphy between $H'[Y_1]$ and $H[Y_1]$. It follows that $H' \sim H$.

□

Proof of Theorem 6. Consider a decomposable tournament T defined on a vertex set V with $\mid V \mid = n \geq 9$, a tournament T' $\{-3\}$-hypomorphic to T. By Corollary 16, T' is (≤ 3)-hypomorphic to T. So, from Corollary 18, $\mathcal{P}(T) = \mathcal{P}(T')$. In particular, if T is strongly connected, then $\widetilde{P}(T) = \widetilde{P}(T')$.

If T is almost transitive, then clearly $T' \sim T$. Let's suppose that T is not almost transitive. For the proof, we distinguish the following two cases.

$*$ T is strongly connected.

In this case, from Proposition 28, $T'/\mathcal{P}(T) = T/\mathcal{P}(T)$ and we can suppose that $\mid \mathcal{P}(T) \mid = 3$ and there exists $X \in \mathcal{P}(T)$ such that $\mid X \mid = n-2$. Let $\mathcal{P}(T) = \{X, \{a\}, \{b\}\}$ where $X \longrightarrow a \longrightarrow b \longrightarrow X$ in T. We verify easily that $T'[X]$ and $T[X]$ are $\{-1,-2,-3\}$-hypomorphic (because T' and T are $\{-3\}$-hypomorphic). We consider the following three cases.

- $T[X]$ is non-strongly connected.

  As the tournaments $T'[X]$ and $T[X]$ are $\{-1\}$-hypomorphic, then they are isomorphic, since the non-strongly connected tournaments with at least 5 vertices are $\{-1\}$-reconstructible [13].

- $T[X]$ is strongly connected and decomposable.

  Let $Q = \mathcal{P}(T[X])$. Assume that there exists $Y \in Q$ such that $\mid Y \mid = \mid X \mid -2$. In this case, we have $\mid Q \mid = 3$ and $T[X]/Q$ is a 3-cycle. Moreover, as $T'[X]$ and $T[X]$ are (≤ 3)-hypomorphic, then by Corollary 18, $\mathcal{P}(T'[X]) = \mathcal{P}(T[X])$ and $T'[X]/\mathcal{P}(T[X]) = T[X]/\mathcal{P}(T[X])$ or $T'[X]/\mathcal{P}(T[X]) = T^*[X]/\mathcal{P}(T[X])$. As in addition, $T'[Y] \sim T[Y]$ (because $T'[X]$ and $T[X]$ are $\{-2\}$-hypomorphic), then $T'[X] \sim T[X]$. Thus, clearly $T' \sim T$.
  Now, suppose that for every $Y \in Q$, $\mid Y \mid < \mid X \mid -2$. In this case, $T[X]$ is not almost transitive. If $\mid X \mid \geq 9$, as $T'[X]$ and $T[X]$ are $\{-3\}$-hypomorphic, then by Proposition 28, $T'[X] \sim T[X]$ and hence, clearly $T' \sim T$. If $\mid X \mid = 7$. As $T'[X]$ and $T[X]$ are $\{-1,-2,-3\}$-hypomorphic, then they are (≤ 6)-hypomorphic, and thus $T'[X] \sim T[X]$ (by [17]). If $\mid X \mid = 8$. As $T'[X]$ and $T[X]$ are $\{-2,-3\}$-hypomorphic, then, by Lemma 31, $T'[X] \sim T[X]$.

- If $T[X]$ is indecomposable.
  In this case, from Theorem 17, $T'[X] = T[X]$ or $T'[X] = T^*[X]$ (because $T'[X]$ and $T[X]$ are (≤ 3)-hypomorphic). If $T'[X] = T[X]$, then clearly $T' \sim T$. If $T'[X] = T^*[X]$, then $T[X]$ is $\{-1,-2,-3\}$-self dual (because $T'[X]$ and $T[X]$ are $\{-1,-2,-3\}$-hypomorphic). We obtain: If $T[X]$ is self dual, then $T'[X] \sim T[X]$ and clearly $T' \sim T$. If $T[X]$ is not self dual, then $T \in C_3(I_{n-2,\{-1,-2,-3\}}) \subset \Omega_n$ and clearly $T' \not\sim T$.

$*$ T is non-strongly connected.

Observe that if T is a transitive tournament, then T' is also transitive (by (≤ 3)-hypomorphy) and the result is obvious. Suppose then that

T is a non-strongly connected tournament which is not transitive. The result follows from the following five facts.

Fact 1. Let X be an element of $\widetilde{\mathcal{P}}(T)$ such that $T[X]$ is strongly connected with $\mid X \mid \geq 3$ and let a be an element of $V \setminus X$. As $\mathcal{P}(T') = \mathcal{P}(T)$, then $T'[X]$ is strongly connected, $X \in \widetilde{P}(T')$ and we have:
$a \longrightarrow X$ in T if and only if $a \longrightarrow X$ in T'.

Indeed :

Let $\{\alpha, \beta, \gamma\}$ be a subset of X such that $T[\{\alpha, \beta, \gamma\}]$ is a 3-cycle and suppose, for example, that $X \longrightarrow a$ in T. From Corollary 15, $n(T, \delta^+; \{\alpha, \beta, a\}) = n(T', \delta^+; \{\alpha, \beta, a\})$. As $n(T, \delta^+; \{\alpha, \beta, a\}) \neq 0$, thus there exists a subset K of V such that $\{\alpha, \beta, a\} \subset K$ and $T'[K] \sim \delta^+$. Hence, $\mid K \cap X \mid = 3$, because otherwise $K \cap X$ is an interval with two elements of $T'[K]$; which contradicts Remark 22. As $a \in V \setminus X$ and $K \setminus \{a\} \subset X$, then $K \setminus \{a\}$ is an interval of $T'[K]$ and thus $T'[K]$ is a diamond of center a. So, $X \longrightarrow a$ in T'.

Fact 2. $\widetilde{\mathcal{P}}(T') = \widetilde{\mathcal{P}}(T)$.

Indeed :

Consider an element Y of $\widetilde{\mathcal{P}}(T)$. We distinguish the following two cases.

- If $T[Y]$ is strongly connected and $\mid Y \mid \geq 3$. In this case, $Y \in \mathcal{P}(T)$. Then $Y \in \mathcal{P}(T')$ and hence, $Y \in \widetilde{\mathcal{P}}(T')$.
- If $T[Y]$ is a transitive. In this case, there exists an element Z of $\widetilde{\mathcal{P}}(T')$ such that $Y \subset Z$, because otherwise, Y admits a partition $\{Y_1, Y_2\}$ such that there is $K \in \mathcal{P}(T')$ with $\mid K \mid \geq 3$, $T'[K]$ is strongly connected and in T' we have $Y_1 \longrightarrow K \longrightarrow Y_2$; which contradicts the Fact 1. While exchanging the roles of T and T', we can hence deduce that $Y = Z$ and then, $Y \in \widetilde{\mathcal{P}}(T')$.

Fact 3. $T'/\widetilde{\mathcal{P}}(T) = T/\widetilde{\mathcal{P}}(T)$.

Indeed:

Proceed by the absurd and suppose that there exist two distinct elements X and Y of $\widetilde{\mathcal{P}}(T)$ such that $X \longrightarrow Y$ in T and $Y \longrightarrow X$ in T'. From the Fact 1, $T[X]$ and $T[Y]$ are transitive. So, X and Y are not consecutive in $T/\widetilde{\mathcal{P}}(T)$. Thus, there exists an element Z of $\widetilde{\mathcal{P}}(T)$ such that $T[Z]$ is strongly connected, $\mid Z \mid \geq 3$ and $X \longrightarrow Z \longrightarrow Y$ in T. So, by Fact 1, $X \longrightarrow Z \longrightarrow Y$ in T' and then $X \longrightarrow Y$ in T'; which is absurd.

Fact 4. If $T \notin \Omega_n$, then for all $X \in \widetilde{\mathcal{P}}(T)$, $T'[X] \sim T[X]$.

Indeed:

Suppose that $T \notin \Omega_n$ and consider an element X of $\widetilde{\mathcal{P}}(T)$. As T and T' are (≤ 3)-hypomorphic, we can assume that $\mid X \mid \geq 4$ and $T[X]$ is strongly connected.

We distinguish the following cases.

- If $\mid X \mid \leq n-3$.
  Consider $H = T[X]$ and $A \subset X$ such that $\mid A \mid = 2$. From Corollary 15, $n(T, H; A) = n(T', H; A)$. As $n(T, H; A) \neq 0$, then $n(T', H; A) \neq 0$. So, there exists a subset K of V such that $A \subset K$ and $T'[K] \sim H$. We have then $K = X$, because otherwise, there exists $Y \in \widetilde{\mathcal{P}}(T) \setminus \{X\}$ such that $K \cap Y \neq \emptyset$ and hence $T'[K]$ is non-strongly connected (because $K \cap X$ is also non empty); which is absurd. So, $T'[X] \sim H = T[X]$.
- If $\mid X \mid = n-1$.
  Let $\{a\} = V \setminus X$ and suppose, for example, that $X \longrightarrow a$ in T. As $\mid X \mid -2 = n-3$, then $T'[X]$ and $T[X]$ are $\{-2\}$-hypomorphic. Now we shall prove that these two tournaments are $\{-3\}$-hypomorphic. For that, consider a subset A of X such that $\mid A \mid = 3$. It is clear that $T' - A$ and $T - A$ are isomorphic and in these two tournaments, we have $(X \setminus A) \longrightarrow a$. So, $T'[X \setminus A] \sim T[X \setminus A]$. Hence, $T'[X]$ and $T[X]$ are $\{-3\}$-hypomorphic. So, $T'[X]$ and $T[X]$ are $\{-2, -3\}$-hypomorphic.
  - If $T[X]$ is decomposable. Let $Q = \mathcal{P}(T[X])$. If there exists $Y \in Q$ such that $\mid Y \mid = \mid X \mid -2$. In this case, $\mid Q \mid = 3$ and $T[X]/Q$ is a 3-cycle. As $\mid Y \mid = n-3$, then $T'[Y] \sim T[Y]$. As in addition, $T'[X]$ and $T[X]$ are (≤ 3)-hypomorphic, then, by Corollary 18, $\mathcal{P}(T'[X]) = Q$ and $T'[X]/Q = T[X]/Q$ or $T'[X]/Q = T^*[X]/Q$. So we deduce immediately that $T'[X] \sim T[X]$. Now suppose that for every $Y \in Q$, $\mid Y \mid < \mid X \mid -2$. In this case, $T[X]$ is not almost transitive. Distinguish the following two cases. If $\mid X \mid \geq 9$. In this case, by applying Proposition 28 for the tournaments $T[X]$ and $T'[X]$, we obtain $T'[X] \sim T[X]$. If $\mid X \mid = 8$. As $T'[X]$ and $T[X]$ are $\{-2, -3\}$-hypomorphic, then, by Lemma 31, $T'[X] \sim T[X]$.
  - If $T[X]$ is indecomposable. We have $T'[X]$ and $T[X]$ are (≤ 3)-hypomorphic. So, by Theorem 17, $T'[X] = T[X]$ or $T'[X] = T^*[X]$. Consider then the case where $T'[X] = T^*[X]$. If $T[X]$ is self dual, then $T'[X] \sim T[X]$. Suppose

hence that $T[X]$ is not self dual. As $T[X]$ is $\{-2,-3\}$-self dual (because $T'[X]$ and $T[X]$ are $\{-2,-3\}$-hypomorphic), then $T \in O_2(I_{n-1,\{-2,-3\}}) \subset \Omega_n$; which is absurd.

- If $\mid X \mid = n-2$.
  Let $\{a,b\} = V \setminus X$ and suppose, for example, that $X \longrightarrow a$ in T. As $\mid X \mid -1 = n-3$, then $T'[X]$ and $T[X]$ are $\{-1\}$-hypomorphic. Now, we shall prove that $T'[X]$ and $T[X]$ are $\{-2,-3\}$-hypomorphic. For that, for every $i \in \{2,3\}$, let A_i be a subset of X such that $\mid A_i \mid = i$. We have $T - (A_2 \cup \{a\}) \sim T' - (A_2 \cup \{a\})$. From Fact 3, $(X \setminus A_2) \longrightarrow b$ in T if and only if $(X \setminus A_2) \longrightarrow b$ in T'. So, $T'[X \setminus A_2] \sim T[X \setminus A_2]$ and thus, $T'[X]$ and $T[X]$ are $\{-2\}$-hypomorphic. Furthermore, as $T' - A_3 \sim T - A_3$, then by Fact 3, we can see that $T'[X \setminus A_3] \sim T[X \setminus A_3]$. So, $T'[X]$ and $T[X]$ are $\{-1,-2,-3\}$-hypomorphic.

  - If $T[X]$ is decomposable. Pose $Q = \mathcal{P}(T[X])$. If there exists $Y \in Q$ such that $\mid Y \mid = \mid X \mid -2$. In this case, $\mid Q \mid = 3$ and $T[X]/Q$ is a 3-cycle. Besides, by $\{-3\}$-hypomorphy, the two subtournaments $T'[Y \cup \{a\}]$ and $T[Y \cup \{a\}]$ are isomorphic (because $\mid Y \cup \{a\} \mid = n-3$). As in addition, $Y \longrightarrow a$ in both these tournaments, then $T'[Y] \sim T[Y]$. As in addition, $T'[X]$ and $T[X]$ are (≤ 3)-hypomorphic, then, by Corollary 18, we can see that $T'[X] \sim T[X]$. Now, suppose that for every $Y \in Q$, $\mid Y \mid < \mid X \mid -2$. In this case, $T[X]$ is not almost transitive. We distinguish the following three cases.
    If $\mid X \mid \geq 9$, we conclude by Proposition 28.
    If $\mid X \mid = 8$, we conclude by Lemma 31.
    If $\mid X \mid = 7$, as $T'[X]$ and $T[X]$ are $\{-1,-2,-3\}$-hypomorphic, then they are (≤ 6)-hypomorphic and hence $T'[X] \sim T[X]$.
  - If $T[X]$ is indecomposable. As $T'[X]$ and $T[X]$ are (≤ 3)-hypomorphic, then from Theorem 17, $T'[X] = T[X]$ or $T'[X] = T^*[X]$. Consider then the case where $T'[X] = T^*[X]$. If $T[X]$ is self dual, then $T'[X] \sim T[X]$. Suppose now that $T[X]$ is not self dual. As in addition $T[X]$ is $\{-1,-2,-3\}$-self dual (because $T'[X]$ and $T[X]$ are $\{-1,-2,-3\}$-hypomorphic), hence the tournament T belongs to $O_3(I_{n-2,\{-1,-2,-3\}})$ and then to Ω_n; which is absurd.

Fact 5. If $T \in \Omega_n$, then $T' \not\sim T$.

This fact is an immediate consequence of facts 2 and 3.

□

We shall now prove Corollary 8. Before that let us observe that Corollary 7 is an immediate consequence of Theorem 6 because each element T of Ω_n (where $n \geq 9$) has an interval X such that $T[X]$ is indecomposable and $\mid X \mid \in \{n-1, n-2\}$.

Proof of Corollary 8. Suppose that there exists an integer $n_0 \geq 7$ such that the indecomposable tournaments with at least n_0 vertices are $\{-3\}$-reconstructible and consider a tournament T with $n \geq n_0 + 2$ vertices. Then, the classes $I_{n-2,\{-3\}}$ and $I_{n-1,\{-3\}}$ are empty. So, the classes $I_{n-2,\{-1,-2,-3\}}$ and $I_{n-1,\{-2,-3\}}$ are empty. Thus, Ω_n is empty and then $T \notin \Omega_n$. By Theorem 6, the tournament T is then $\{-3\}$-reconstructible.

□

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