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Asymptotic study of the convective parametric instability in Hele-Shaw cell

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This paper concerns a linear study of the convective parametric instability in the case of a Newtonian fluid confined in a Hele-Shaw cell and submitted to a vertical periodic motion. The gradient of temperature, applied to the fluid layer, is either in the same direction that gravity or in the opposite one. An asymptotic analysis shows that the Hele-Shaw approximation leads to two linear formulations depending on the order of magnitude of the Prandtl number. For these two asymptotic cases, the convective threshold is determined. It turns out that in the Hele-Shaw geometrical configuration, parametric oscillations have no influence on the criterion of stability when the Prandtl number is in the order of the unity or very superior to the unity. However, when the Prandtl number is small than unity, the parametric oscillations can affect the convective instability threshold. © *2000 American Institute of Physics.* [S1070-6631(00)00302-0]

I. INTRODUCTION

Consider a horizontal layer of a Newtonian fluid submitted to a gradient of temperature ∇T parallel to the gravitational acceleration $\mathbf{g}$. If the directions of ∇T and $\mathbf{g}$ coincide, the Rayleigh number is positive and then the corresponding configuration is an unstable equilibrium: Beyond a critical value of the Rayleigh number, a convective flow is established in a cellular structural form. In the case where ∇T has an opposite direction that $\mathbf{g}$, the Rayleigh number is negative, and the corresponding configuration is a stable equilibrium. This last configuration may be destabilized by a gravity modulation which can be realized by oscillating vertically the fluid layer and its frontiers. Gresho and Sani[1] have treated the case of a fluid layer confined between two rigid horizontal boundaries. They have studied the influence of the gravitational modulation on the convective threshold of the stable and unstable equilibrium configurations. By means of a Galerkin method truncated to the first order, Gresho and Sani[1] have reduced the governing linear system, corresponding to the bidimensional perturbations with respect to the rest state, to the Mathieu equation. The stability analysis has been carried out qualitatively by exploiting the ''Mathieu stability charts.'' Similar treatment on the topic was reviewed in details in Gershuni and Zhukovitskii.[2] Later, Biringen and Peltier[3] have extended the linear bidimensional problem of Gresho and Sani to a nonlinear tridimensional one. In this work, they have confirmed the bidimensional character of Gresho and Sani results at the convective threshold. Recently, Clever *et al.*[4] have investigated the effects of gravitational modulation on the bidimensional convective threshold and constructed nonlinear solutions via Galerkin method. They have also treated the tridimensional oscillatory convection under gravity modulation.[5]

Without being exhaustive, we quote other works on the gravitational modulation. Wadih and Roux[6] have considered the case of a fluid occupying a cylindrical cavity of infinite length and submitted to a negative gradient of temperature maintained in the upward direction. They have also studied the effect of the gravitational modulation on the convective threshold. Braveman and Oron[7] have carried out an analytical-numerical study of stability in the case of a fluid layer performing high-frequency oscillations with vertical and horizontal components. Using an averaging method, they have shown that when the boundary conditions are not symmetric, the instability of the equilibrium is oscillatory. These authors have completed this study by a weakly nonlinear analysis where an amplitude equation has been derived and studied analytically and numerically.[8]

In contrast to the gravitational modulation, other works have been devoted to the modulation of the temperature imposed on the frontier. Venezian[9] has considered the case where the vertical gradient of temperature, imposed to the fluid layer, possesses a stationary component together with a time-periodically one having small amplitude. Performing a linear stability analysis in the case of free–free horizontal boundaries, he has shown that the modulation of the frontier temperature may stabilize or destabilize the equilibrium state. Roppo *et al.*[10] have completed this linear study by a

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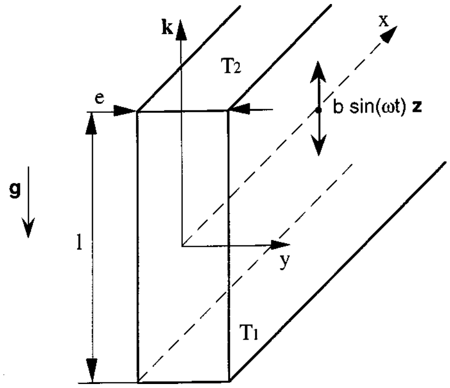

FIG. 1. Scheme arrangement of Hele-Shaw cell.

weakly nonlinear analysis. A similar problem to the one by Venezian[9] has been considered previously by Gershuni and Zhukhovitski[11] taking into account that fluctuations of the temperature obey a rectangular law. This problem (free–free horizontal boundaries) has been studied by other authors. For instance, Rosenblat and Herbert[12] have provided an asymptotic solution in the case of low modulation frequency with modulation amplitude not small, Yih and Li[13] have used a Galerkin method to investigate the stability in the case where the gradient of temperature is symmetrical.

The purpose of this work is to analyze the influence of the gravitational modulation on the instability threshold in the case of a Hele-Shaw geometrical configuration. In this configuration, a viscous fluid, in a narrow slot between vertical parallel plates, is submitted to a constant gradient of temperature either in the same direction that **g** or in the opposite one.

The analogy between motion in Hele-Shaw cell and motion in porous medium, for weak Rayleigh numbers, has frequently been used to simulate porous convection.[14–16] This analogy has served especially to detect experimentally the critical Rayleigh number corresponding to the onset of convection.

In the present paper, a dimensional and asymptotic analysis allow us to distinguish between two linear formulations. Each one of these formulations depends on the order of magnitude of the Prandtl number. The first formulation corresponds to $\mathrm{Pr}=O(1)$ or $\mathrm{Pr}\gg 1$ while the second one corresponds to $\mathrm{Pr}=O(\epsilon^2)$, where ϵ designates the aspect ratio of the cell. Therefore, we determine the criterion of stability for these two asymptotic cases.

II. FORMULATION

Consider a Newtonian fluid confined in a horizontal Hele-Shaw cell of infinite extent in the x direction (see Fig. 1). Denote by l the height of the cell, e the distance between the vertical planes and $\epsilon=e/l\ll 1$ the aspect ratio of the cell; the values $y=\pm e/2$ and $z=0$, l correspond to the boundaries of the cell. We assume that the fluid confined in the cell is bounded vertically by two thermally insulating walls and horizontally by two perfect heat conducting plates having respectively constant temperature T_1 and T_2. Suppose that the Hele-Shaw cell is submitted to an oscillatory motion according to the law of displacement $b\sin(\omega t)\mathbf{k}$, where b and ω designate, respectively, the displacement amplitude and the dimensional angular frequency of the oscillatory motion; $\mathbf{k}$ is the unit vector upward. Therefore, the fluid layer is submitted to two volumic forces: The gravitational force field $\rho\mathbf{g}$ and the oscillatory force one $-\rho b\omega^2\sin(\omega t)\mathbf{k}$. The equilibrium of the fluid layer corresponds to a rest state with a conductive regime. Under these assumptions, the linear system of the governing equations, corresponding to the perturbation of the equilibrium state, is given by the following Navier–Stokes equations in the Boussinesq approximation

$$\operatorname{div}\mathbf{V}=0, \tag{1}$$

$$\rho\frac{\partial\mathbf{V}}{\partial t}=-\operatorname{grad}P+\rho\nu\Delta V+\rho\beta T[g+b\omega^2\sin(\omega t)]\mathbf{k}, \tag{2}$$

$$\frac{\partial T}{\partial t}=\frac{T_1-T_2}{l}w+\kappa\Delta T, \tag{3}$$

where ρ, β, ν, and κ designate, respectively, the density, the coefficient of thermal dilatation, the kinematic viscosity, and the thermal diffusivity of the fluid.

To introduce a perturbation parameter involving only the aspect ratio of the cell, ϵ, we perform a dimensional analysis by means of an appropriate choice of scales used in convection problems in Hele-Shaw cell.[17,18] Thus, the time is scaled by l^2/κ, the coordinates (x,y,z) are scaled by (l,e,l), the velocity field $\mathbf{V}(u,v,w)$ is scaled by $(\kappa/l,\ e\kappa/l^2,\ \kappa/l)$, and the pressure and temperature are, respectively, scaled by $\rho\nu\kappa/e^2$ and (T_1-T_2). Hence, the linear system of Eqs. (1)–(3) is written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{4}$$

$$\epsilon^2\,\mathrm{Pr}^{-1}\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\epsilon^2\Delta_2 u+\frac{\partial^2 u}{\partial y^2}, \tag{5}$$

$$\epsilon^4\,\mathrm{Pr}^{-1}\frac{\partial v}{\partial t}=-\frac{\partial p}{\partial y}+\epsilon^4\Delta_2 v+\epsilon^2\frac{\partial^2 v}{\partial y^2}, \tag{6}$$

$$\epsilon^2\,\mathrm{Pr}^{-1}\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+\epsilon^2\Delta_2 w+\frac{\partial^2 w}{\partial y^2}+\mathrm{Ra}[1+\alpha\sin(\Omega t)]T, \tag{7}$$

$$\epsilon^2\frac{\partial T}{\partial t}=\epsilon^2 w+\epsilon^2\Delta_2 T+\frac{\partial^2 T}{\partial y^2}, \tag{8}$$

where $\Delta_2=\partial^2/\partial x^2+\partial^2/\partial z^2$, $\mathrm{Ra}=\beta g\Delta T e^2 l/\nu\kappa$ is the gravitational Rayleigh number of the cell, $\Omega=\omega l^2/\kappa$ is a nondimensional frequency, and $\alpha=b\omega^2/g$ represents the amplitude's ratio of the oscillatory motion acceleration and the gravity one.

Moreover, on the vertical walls, the boundary conditions are: $\mathbf{V}=0$ and $\partial T/\partial y=0$ at $y=\pm\frac{1}{2}$. Note that the boundary conditions on the horizontal plates will be discussed below.

III. ASYMPTOTIC STUDY AND STABILITY

It is obvious that in the Hele-Shaw approximation, the order of magnitude of the Prandtl number Pr in the system of

Eqs. (4)–(8) must be estimated to be able to drop terms of the order ϵ^2. In this situation, two different formulations depending on the order of magnitude of the Prandtl number can be distinguished.

A. First formulation $\mathrm{Pr}=O(1)$ or $\mathrm{Pr}\gg 1$

In this formulation, a first approximation is obtained from the system of Eqs. (4)–(8) by setting $\epsilon^2=0$. Remark that in this situation, the term $\partial \mathbf{V}/\partial t$ disappears. Let us denote by u_o, v_o, w_o, p_o, and T_o the solution of such approximation. From Eq. (6), the pressure is independent of y. Also, from Eq. (8) and using the adiabatic condition, the temperature T_o is also independent of y. Thus, systems (4)–(8) become

$$\frac{\partial u_o}{\partial x}+\frac{\partial v_o}{\partial y}+\frac{\partial w_o}{\partial z}=0, \tag{9}$$

$$\frac{\partial^2 u_o}{\partial y^2}=\frac{\partial p_o}{\partial x}, \tag{10}$$

$$\frac{\partial^2 w_o}{\partial y^2}=\frac{\partial p_o}{\partial z}-\mathrm{Ra}[1+\alpha\sin(\Omega t)]T_o. \tag{11}$$

Integrating Eqs. (10) and (11), we obtain u_o and w_o. Substituting u_o and w_o into the continuity Eq. (9) and integrating, v_o is determined. The solution v_o which satisfies the boundary condition $v_o=0$ at $y=\pm\frac{1}{2}$ is $v_o=0$.

Clearly, at first order ($\epsilon^2=0$), Eqs. (9)–(11) cannot be coupled to the energy equation [Eq. (8)]. The situation corresponds to a pseudo-singular perturbation. Therefore, the energy equation [Eq. (8)] is exploited at the order ϵ^2 by using the accurate expansion

$$w=w_o+\epsilon^2 w_1, \tag{12a}$$

$$T=T_o+\epsilon^2 T_1. \tag{12b}$$

Inserting expression (12) into Eq. (8) and keeping only terms of order ϵ^2, one obtains

$$\frac{\partial T_o}{\partial t}=w_o+\Delta_2 T_o+\frac{\partial^2 T_1}{\partial y^2}, \tag{13}$$

in which T_1 must also verify the adiabatic condition on the vertical walls: $\partial T_1/\partial y=0$ at $y=\pm\frac{1}{2}$. In contrast to the original systems (4)–(8), which needs six boundary conditions at the horizontal walls, the system to the first-order (9)–(11) and (13), resulting from the Hele–Shaw approximation, requires only the relevant four boundary conditions

$$w_o=T_o=0 \quad \text{at } z=\pm\tfrac{1}{2}.$$

To perform a stability analysis, we seek the solution of systems (9)–(11) and (13) in term of normal modes as

$$u_o=\pi g(t)(y^2-\tfrac{1}{4})\sin(\pi z)e^{iqx}, \tag{14}$$

$$w_o=iqg(t)(y^2-\tfrac{1}{4})\cos(\pi z)e^{iqx}, \tag{15}$$

$$T_o=iqf(t)\cos(\pi z)e^{iqx}, \tag{16}$$

where q denotes the wave number, $f(t)$ and $g(t)$ designate, respectively, the amplitudes of the temperature and the velocity field. Using the above assumptions, systems (9)–(11) and (13) is reduced to the differential equation

$$\dot{f}+\frac{h}{12}[R_o-\mathrm{Ra}(1+\alpha\sin(\Omega t))]f=0, \tag{17}$$

where $h=\lambda/(\lambda+1)$, $\lambda=q^2/\pi^2$ and $R_o=12\pi^2(\lambda+1)^2/\lambda$; R_o is the Rayleigh number corresponding to the marginal stability of the unmodulate case.[19] The solution of Eq. (17) is given by

$$f(t)=f_o e^{-(h/12)[(R_o-\mathrm{Ra})t+(\alpha\mathrm{Ra}/\Omega)\cos(\Omega t)]}, \tag{18}$$

where f_o is an arbitrary constant depending on the initial conditions.

If the fluid layer is heated from below (i.e., $\mathrm{Ra}>0$), we can see from Eq. (18) that the stability criterion is $\mathrm{Ra}\leqslant R_o$. This criterion is the same that the one corresponding to the unmodulate case: $\mathrm{Ra}_c=48\pi^2$ and $q_c=\pi$.[19] Nevertheless, at the onset of convection, the amplitudes of temperature and velocity field are periodic and have the same frequency that the parametric excitation. These amplitudes are given by

$$f(t)=f_o e^{-2\pi^2\,\mathrm{Fr}\,\Omega\,\cos(\Omega t)}, \tag{19}$$

$$g(t)=-12\pi^2 f_o[1+\mathrm{Fr}\,\Omega^2\sin(\Omega t)]e^{-2\pi^2\,\mathrm{Fr}\,\Omega\,\cos(\Omega t)}, \tag{20}$$

where $\mathrm{Fr}=\alpha\Omega^{-2}=b\kappa^2/gl^4$ represents a sort of Froude number $\{=[(\kappa/l)^2/gl](b/l)\}$. In this case, the parametric excitation has neither a stabilizing effect nor a destabilizing one.

On the other hand, if the fluid layer is heated from above (i.e., $\mathrm{Ra}<0$), the rest state of the fluid layer cannot be destabilized.

Hence, we can conclude in this first formulation that if the fluid has a Prandtl number $\mathrm{Pr}=O(1)$ or $\mathrm{Pr}\gg 1$, the oscillations of the cell cannot generate convective parametric instabilities. The physical reason for that is related to the large friction at the vertical walls which leads to the suppression of inertial effects. Quite the same situation takes place in the case of a porous medium when the Darcy law is assumed for the resistance force.

B. Second formulation $\mathrm{Pr}=0(\epsilon^2)$

In this formulation the term $\partial\mathbf{V}/\partial t$ remains in Eqs. (5) and (7) after making the Hele-Shaw approximation ($\epsilon^2=0$). Hence, setting $\mathrm{Pr}=\epsilon^2\,\mathrm{Pr}^*$ where $\mathrm{Pr}^*=O(1)$, systems (4)–(8) become

$$\frac{\partial u_o}{\partial x}+\frac{\partial v_o}{\partial y}+\frac{\partial w_o}{\partial z}=0, \tag{21}$$

$$\mathrm{Pr}^{*-1}\frac{\partial u_o}{\partial t}=-\frac{\partial p_o}{\partial x}+\frac{\partial^2 u_o}{\partial y^2}, \tag{22}$$

$$\mathrm{Pr}^{*-1}\frac{\partial w_o}{\partial t}=-\frac{\partial p_o}{\partial z}+\frac{\partial^2 w_o}{\partial y^2}+\mathrm{Ra}[1+\alpha\sin(\Omega t)]T_o, \tag{23}$$

$$\frac{\partial T_o}{\partial t}=w_o+\Delta_2 T_o+\frac{\partial^2 T_1}{\partial y^2}, \tag{24}$$

where $p_o=p_o(x,z)$ and $T_o=T_o(x,z)$. As previously, the energy equation [Eq. (24)] is obtained at the order ϵ^2, in which the term T_1 verifies the adiabatic condition on the vertical walls: $\partial T_1/\partial y=0$ at $y=\pm\frac{1}{2}$. The solution of systems (21)–(24) can be sought in forms (14)–(16). Using Eq. (21) we find that $v_o=0$ and then, after substituting expressions (14)–(16) into Eqs. (22)–(24) and averaging with respect to y, we obtain the differential equation:

$$\ddot{f}+2p\dot{f}+h\,\mathrm{Pr}^*[R_o-\mathrm{Ra}(1+\alpha\sin(\Omega t))]f=0, \tag{25}$$

where $2p=\pi^2+q^2+12\,\mathrm{Pr}^*$.

Now we shall to analyze the stability of the equilibrium state using the amplitude equation of temperature (25). The change of variables $f(t)=e^{-pt}F(t)$ and $2\tau=\Omega t-\pi/2$ reduces Eq. (25) to the well-known Mathieu equation

$$\ddot{F}+[A-2B\cos(2\tau)]F=0, \tag{26}$$

where A and B are given by expressions

$$A=\frac{4\lambda\,\mathrm{Pr}^*(R_N-\mathrm{Ra})}{(1+\lambda)\Omega^2},\quad B=2\,\mathrm{Fr}\,\mathrm{Pr}^*\frac{\lambda}{1+\lambda}\mathrm{Ra}$$

and

$$R_N=-R_o\frac{[1-\mathrm{Pr}^*G(\lambda)]^2}{4G(\lambda)\mathrm{Pr}^*},\quad G(\lambda)=\frac{12}{\pi^2(1+\lambda)},$$
$$\lambda=\frac{q^2}{\pi^2}. \tag{27}$$

Following the Floquet Theory, the general solution of Mathieu equation [Eq. (26)] is of the form

$$F(\tau)=e^{\mu\tau}P(\tau), \tag{28}$$

where μ is the Floquet exponent and $P(\tau)$ is a periodic function with period π or 2π. The solutions of Eq. (25) are then given by

$$f(t)=e^{-pt}F(t)=e^{(\mu\Omega/2-p)t}G(t), \tag{29}$$

where $G(t)$ is a periodic function with period π or 2π. From Eq. (29), we can see that the the criterion of stability is

$$\frac{\mu\Omega}{2}\leqslant p. \tag{30}$$

Hereafter, we focus our attention on the marginal stability condition, corresponding to periodic solutions of period π or 2π, given by

$$\frac{\mu\Omega}{2}=p. \tag{31}$$

The solutions of Eq. (26) can be expressed in the form[20]

$$F_\pi=e^{\mu\tau}\sum_{-\infty}^{+\infty}a_ne^{2in\tau}, \tag{32}$$

$$F_{2\pi}=e^{\mu\tau}\sum_{-\infty}^{+\infty}a_ne^{i(2n+1)\tau}. \tag{33}$$

The expression (32) corresponds to the synchronous solutions having the same frequency that the parametric excitation while expression (33) corresponds to the subharmonic ones. First, consider the synchronous solutions of Eq. (26). Substitution of expression (32) in Eq. (26) leads to the following linear system:

$$\xi_na_{n-1}+a_n+\xi_na_{n+1}=0,\quad n=\cdots-2,-1,0,1,2\cdots, \tag{34}$$

where $\xi_n(\mu)=B/[(2n-i\mu)^2-A]$. For non vanishing solutions, the determinant of the matrix in Eq. (34) must vanish. One obtains

$$\Delta(i\mu)=\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \xi_{-1} & 1 & \xi_{-1} & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \xi_1 & 1 & \xi_1 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}=0, \tag{35}$$

where $\Delta(i\mu)$ is the Hill determinant. Equation (35) may be written as[20]

$$Ch(\mu\pi)=1-2\Delta(0)\mathrm{Sin}^2\left(\frac{\pi\sqrt{A}}{2}\right), \tag{36}$$

where

$$\Delta(0)=\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \xi_2 & 1 & \xi_2 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & \xi_1 & 1 & \xi_1 & 0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & \xi_1 & 1 & \xi_1 & 0 & \cdot \\ \cdot & 0 & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{37}$$

The determinant $\Delta(0)$ is defined from Eq. (35) for $\mu=0$ and hence functions $\xi_n(\mu)$ are actually $\xi_n(0)$. It should be noticed that the advantage of the transformation (36) is that $\Delta(0)$ can be computed from recurrence relations between the determinants of orders differing by two at each step, starting with the mid-determinant[21]

$$\Delta_0=1,\quad \Delta_1=1-2\xi_0\xi_1,$$
$$\Delta_2=(1-\xi_1\xi_2)^2-2\xi_0\xi_1(1-\xi_1\xi_2), \tag{38a}$$
$$\Delta_{n+2}=(1-\xi_{n+1}\xi_{n+2})\Delta_{n+1}-\xi_{n+1}\xi_{n+2}(1-\xi_{n+1}\xi_{n+2})\Delta_n+\xi_n^2\xi_{n+1}^3\xi_{n+2}\Delta_{n-1}. \tag{38b}$$

Convergence of Eq. (37) is rapid enough to produce data to two or three decimals in a very short time.

Let us discuss the following cases. If the fluid layer is heated from below (Ra>0), then the constant A given by Eq. (27) is negative. Consequently, Eq. (36) becomes

$$Ch(\mu\pi)=1+2\Delta(0)\mathrm{Sinh}^2\left(\frac{\pi\sqrt{-A}}{2}\right). \tag{39}$$

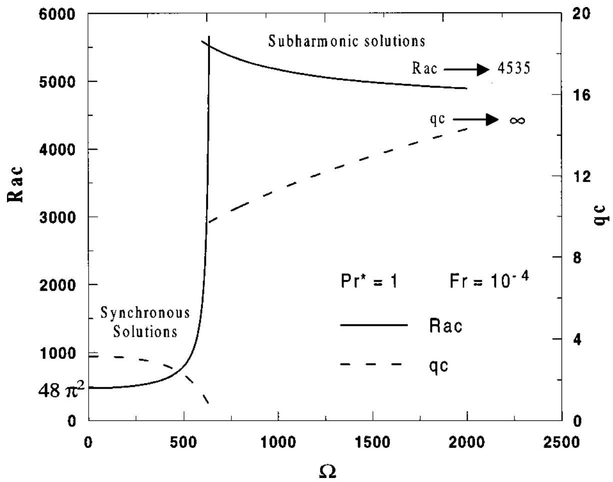

FIG. 2. Heated from below. Evolution of the critical Rayleigh number Ra_c and wave number q_c as a function of the nondimensional frequency Ω.

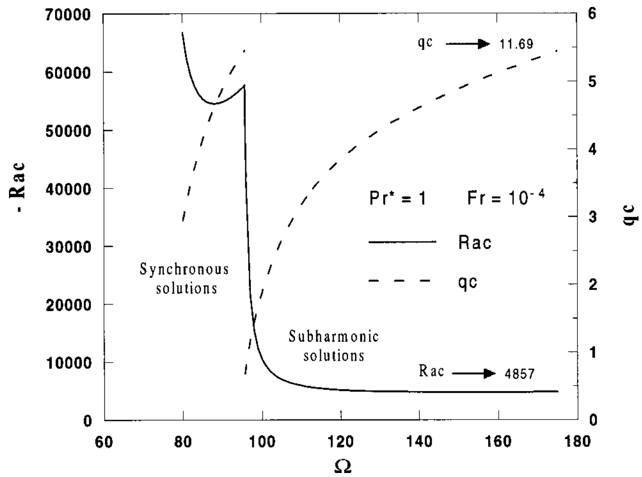

FIG. 3. Heated from above. $\mathrm{Ra}<R_N$. Evolution of the critical Rayleigh number Ra_c and wave number q_c as a function of the nondimensional frequency Ω.

If the fluid layer is heated from above ($\mathrm{Ra}<0$), then the constant A may be either positive for $R_N>\mathrm{Ra}$, or negative for $R_N<\mathrm{Ra}$. We can use Eq. (36) or Eq. (39) depending of the sign of the constant A.

Similarly, for subharmonic solutions, one obtains an equivalent characteristic equation that Eq. (39)

$$Ch(\mu\pi)=-1+2\Delta(0)\,\mathrm{Sinh}^2\left(\frac{\pi\sqrt{-A}}{2}\right). \tag{40}$$

Note that both Eqs. (36) and (40) relate the effective Prandtl number Pr*, the Froude number Fr, the Rayleigh number Ra, the wave number q, the nondimensional frequency Ω, and the Floquet exponent μ. In order to investigate the marginal stability curves, these Eqs. (36) and (40) are solved numerically for fixed Prandtl and Froude numbers. We first determine numerically the marginal stability curves for different values of the frequency Ω, to obtain the corresponding critical Rayleigh numbers Ra_c and wave numbers q_c. To obtain marginal stability curve in the plane Ra vs q, we proceed as follows. For a fixed Ω and a fixed wave number q_i, we first calculate the Floquet exponent μ_i using Eq. (31), and then we determine the corresponding value of $(\mathrm{Ra})_i$ verifying the synchronous characteristic equation [Eq. (36)] or the subharmonic one Eq. (40). Hereafter, we plot the curves corresponding to the critical Rayleigh and wave numbers versus the frequency Ω. Each one of these critical curves, represents the minimum of the two modes of solutions (synchronous and subharmonic) in term of the critical Rayleigh number. Indeed, the instability is caused by the smallest value of the critical Rayleigh number. Therefore, the critical curves consist of two regions, one corresponds to synchronous solutions and the other is related to the subharmonic ones.

Figure 2 illustrates the results of the case where the fluid is heated from below and for values of effective Prandtl number and Froude number $\mathrm{Pr}^*=1$ and $\mathrm{Fr}=10^{-4}$. Near $\Omega=0$, the critical Rayleigh and wave numbers tend, respectively, as expected, to the values of the unmodulate case, namely $\mathrm{Ra}_c=48\,\pi^2$ and $q_c=\pi$.[19] Furthermore, one can see that if Ω increases from zero, the critical Rayleigh number, corresponding to the onset of synchronous solutions, increases until a certain frequency is reached at which the onset is in the form of subharmonic solutions. The crossover frequency between the synchronous and subharmonic solutions, depending on the Froude number Fr and on the effective Prandtl number Pr*, is $\Omega_t=636$. More precisely, for $\Omega<\Omega_t$, values of subharmonic solutions are bigger than the synchcronous ones; also, for $\Omega>\Omega_t$, values of synchronous solutions are bigger than the subharmonic ones. In each case, only the smallest value is significant. For the sake of clarification, we have plotted some data showing such intersection between the synchronous and subharmonic modes.

As Ω increases further beyond Ω_t, the critical Rayleigh number Ra_c corresponding to the onset of subharmonic solutions decreases to reach an asymptotic value $\mathrm{Ra}_c=4535$. The evolution of the wave number q_c, as shown in Fig. 2, gives rise to a jump phenomenon when crossing the value Ω_t. The wave number corresponding to synchronous solutions decreases from $q_c=\pi$ to $q_c=0.664$, while the one corresponding to the subharmonic solutions increases from $q_c=9.7$ to infinity.

In Fig. 3, we present the results corresponding to the case of a fluid layer heated from above for $\mathrm{Ra}<R_N$, $\mathrm{Pr}^*=1$, and $\mathrm{Fr}=10^{-4}$. We can see that the zone of synchronous solutions is very small in contrast to subharmonic zone. In the synchronous region, as Ω tends to zero, the critical Rayleigh number increases to high values, while the critical wave number increases with Ω to reach the value $q_c=5.4$ at $\Omega_t=95$. In the subharmonic region, the critical Rayleigh number decreases to reach the asymptotic value $\mathrm{Ra}_c=4857$ and the critical wave number increases from value $q_c=2.6$ to reach the asymptotic value $q_c=11.7$. As in the first case, a jump phenomenon of the wave number at $\Omega_t=95$ can be observed.

In Fig. 4 we show, for different values of the Froude number and for $\mathrm{Pr}^*=1$, the evolution of the critical Rayleigh number as a function of the frequency in the case where the fluid layer is heated from below. Note that the zone corresponding to the synchronous solutions narrows dramatically

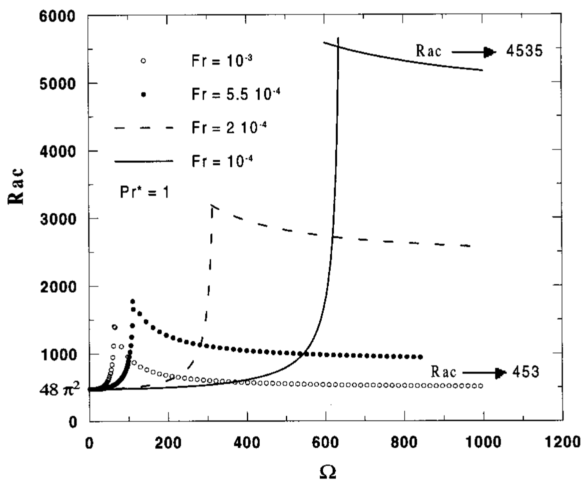

FIG. 4. Heated from below. Evolution of the critical Rayleigh number Ra_c as a function of the nondimensional frequency Ω for different values of Froude number.

as the Froude number increases. In this zone, the parametric excitation has always a stabilizing effect. The subharmonic region widens as the Froude number increases. Here the asymptotic values of the critical Rayleigh number decrease as the Froude number increases. We point out that in this region the parametric excitation may have a destabilizing effect for high frequencies and large Froude numbers related to the amplitude of oscillations. For example, if $Fr=10^{-3}$, the asymptotic value of the critical Rayleigh number, $Ra_c=453$, is less than the one corresponding to the unmodulate case $48\pi^2$. Here, we have a destabilizing effect.

Finally, Fig. 5 illustrates, for $Fr=10^{-4}$ and $\Omega=100$, the dependence of the critical Rayleigh number Ra_c on the effective Prandtl number Pr^*. It is worth noting that for these parameters only the synchronous solutions exist. For instance, if $\epsilon=0.1$, we have $Pr=1$ for $Pr^*=100$ and $Pr\gg 1$ for $Pr^*\gg 100$. It can be seen from Fig. 5 that beyond the value $Pr^*=100$, the critical Rayleigh number tends, as expected, to the critical one of the unmodulate case, $Ra_c=48\,\pi^2$. Therefore, this result verifies that of the first asymptotic case and shows the transition between the two limiting cases.

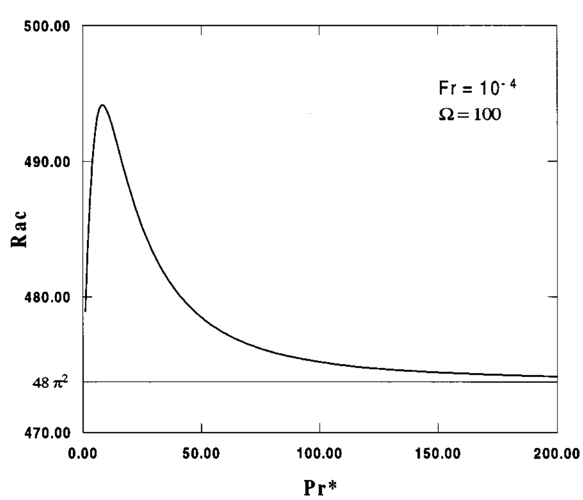

FIG. 5. Heated from below. Evolution of the critical Rayleigh number Ra_c as a function of the effective Prandtl number Pr^* for $\Omega=100$.

IV. SUMMARY

In this work, we have studied the influence of the gravitational modulation on the instability threshold for a fluid layer confined in a Hele-Shaw cell. The fluid layer is submitted to a constant gradient of temperature in the same direction that gravity or in the opposite one. An appropriate choice of the characteristic magnitudes related to the convection problems in Hele–Shaw cell together with an asymptotic analysis provide two different linear formulations. Each formulation depends on the magnitude order of the Prandtl number. We have shown that if the Prandtl number is in the order of the unity or very large than the unity, parametric oscillations have no stabilizing or destabilizing effect. In this formulation, we have the same criterion of stability, in term of the critical Rayleigh number Ra_c and wave number q_c, that the one of the unmodulate case. Nevertheless, at the onset of convection, the amplitudes of the velocity field and temperature, given analytically, are periodic and have the same frequency that the parametric oscillations.

In contrast to the first asymptotic formulation $Pr=O(1)$ or $Pr\gg 1$, the governing system of equations, corresponding to the second formulation $Pr=O(\epsilon^2)$, is reduced to the Mathieu equation. More precisely, the damping effect of the vertical walls can be weakened and the convective parametric instability can occur in liquids with weak Prandtl number as liquids metals. In the geometrical Hele-Shaw configuration, the Mathieu equation is derived directly from the governing system by using normal modes and may be considered as a good approximation to the second formulation. Therefore, the Floquet theory can be applied to determine a simple criterion of stability. It should be noticed that for the numerical determination of the stability criterion, we have performed an alternative method to that by Gresho and Sani.[1] The method developed in this work, consists in exploiting directly the Hill determinant.

In practice, the Hele-Shaw cells allow to realize a simulation of flows in porous medium under certain conditions.[22,23] Therefore, this linear study can serve as model to investigate the influence of the gravitational modulation on the threshold of convective instability in porous medium.

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