Electronic Journal of Differential Equations , Vol . 2001 (2001) , No . 31 , pp . 1 – 2 0 .
ISSN : 1 72 - 6691 . URL : http : / / ej de . math . swt . edu or http : / / ej de . math . unt . edu
ftp ejde . math . swt . edu (login : ftp)

POTENTIAL THEORY FOR QUASILINIEAR ELLIPTIC EQUATIONS

A . BAALAL & A . BOUKRICHA

DEDICATED TO PROF . WOLFHARD HANSEN ON HIS 60 TH BIRTHDAY

ABSTRACT . We discuss the potential theory associated with the quasilinear elliptic equation

$$-\mathrm{div}(\mathcal{A}(x,\nabla u))+\mathcal{B}(x,u)=0.$$

We study the validity of Bauer convergence property , the Brelot convergence property . We discuss the validity of the Keller - Osserman property and the existence of Evans functions .

1 . INTRODUCTION

This paper is devoted to a study of the quasilinear elliptic equation

$$-\mathrm{div}(\mathcal{A}(x,\nabla u))+\mathcal{B}(x,u)=0, \tag{1.1}$$

where $\mathcal{A}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ and $\mathcal{B}:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ are Carath é odory functions satisfying the structure conditions given in Assumptions (I) , (A 1) , (A 2) , (A 3) , and (M) below . In particular we are interested in the potential theory , the degeneracy of the sheaf of continuous solutions and the existence of Evans functions for the equation (1 . 1) .

Equation of the same type as (1 . 1) were investigated in earlier years in many interesting papers , [1 9 , 20 , 1 5 , 1 8] . An axiomatic potential theory associated with the equation div $(\mathcal{A}(x,\nabla u)) = 0$ was recently introduced and discussed in [1 0] . These axiomatic setting are illustrated by the study of the $p-$ Laplace equation $\Delta_p u=$ div $(\mid\nabla u\mid^{p-2}\nabla u)$ obtained by $\mathcal{A}(x,\xi)=\mid\xi\mid^{p-2}\xi$ for every $x\in\mathbb{R}^d$ and $\xi\in\mathbb{R}^d$. We have $\Delta_2=\Delta$ where Δ, the Laplace operator on $\mathbb{R}^d$.

Our paper is organized as follows : In the second section we introduce the ba - sic notation . In the third section we present the structure conditions needed for the mappings $\mathcal{A}$ and $\mathcal{B}$ in order to consider the equation (1 . 1) . We then use the variational inequality to prove the solvability of the variational Dirichlet problem related to (1 . 1) . In section 4 we prove a comparison principle for supersolutions and subsolutions , existence and uniqueness of the Dirichlet problem related to the sheaf $\mathcal{H}$ of continuous solutions of (1 . 1) , as well as the existence of a basis of regular sets

1 99 1 *Mathematics Subject Classification* . 3 1 C 1 5 , 35 J 60 .
Key words and phrases . Quasilinear elliptic equation , Convergence property , Keller - Osserman property , Evans functions .

Submitted October 24 , 2000 . Published May 7 , 200 1 .
Supported by Grant E 2 / C 1 5 from the Tunisian Ministry of Higher Education .

stable by intersection . In the fifth section we discuss the potential theory associated with equation (1 . 1) , prove that the harmonic sheaf $\mathcal{H}$ of solutions of (1 . 1) satisfies the Bauer convergence property , then introduce the presheaves of hyper - harmonic functions $*_{\mathcal{H}}$ and of hypoharmonic functions $*^{\mathcal{H}}$ and prove a comparison principle . In the sixth section we prove , using the obstacle problem , that $*_{\mathcal{H}}$ and $*^{\mathcal{H}}$ are sheaves . In the seventh section we study the degeneracy of the sheaf $\mathcal{H}$; we are not able to prove that the sheaf $\mathcal{H}$ is non degenerate even if we have the following Harnack inequality [1 9 , 20 , 1 8 , 4] :

For every open domain U in $\mathbb{R}^d$ and every compact subset K of U the re exists two non - negative constants c_1 and c_2 such that for every $h \in \mathcal{H}^+(U)$,

$$\sup_K h \leqslant c_1 \inf_K h + c_2.$$

Let U be an open subset of $\mathbb{R}^d, d \geqslant 1$ and α a positive real number , let $0 < \varepsilon < 1$

d

and b be a non - negative function in $L^{p-\varepsilon}_{\text{loc}}(\mathbb{R}^d)$. For every open U we consider the set $\mathcal{H}_\alpha(U)$ of all functions $u \in \mathcal{W}^{1,p}_{\text{loc}}(U) \cap \mathcal{C}(U)$ which are solutions of the equation (1 . 1) with $\mathcal{B}(x, \zeta) = b(x)$ sgn $(\zeta) \mid \zeta \mid^\alpha$, then $(\mathbb{R}^d, \mathcal{H}_\alpha)$ is a nonlinear Bauer space . In particular $\mathcal{H}_\alpha$ is non degenerate on $\mathbb{R}^d$. For $\alpha < p - 1$, the Harnack inequality and the Brelot convergence property are valid , but in contrast to the linear and quasilinear theory (see e . g . [10])$(\mathbb{R}^d, \mathcal{H}_\alpha)$ is not elliptic in the sense of Definition 7 . 1 . In the eighth section , we define , as in [5] , regular Evans functions u tending to the infinity (or exploding) at the regular boundary points of $U.$ We assume that $\mathcal{A}$ satisfies the following supplementary derivability and homogeneity conditions :

- For every $x_0 \in \mathbb{R}^d$, the function F from $\mathbb{R}^d$ to $\mathbb{R}^d$ defined by $F(x) = \mathcal{A}(x, x - x_0)$ is differentiable and div F is lo cally (essentially) bounded .
- $\mathcal{A}(x, \lambda\xi) = \lambda \mid \lambda \mid^{p-2} \mathcal{A}(x, \xi)$ for every $\lambda \in \mathbb{R}$ and every $x, \xi \in \mathbb{R}^d$.

These conditions are satisfied in the particular case of the $p-$ Laplace operator with $p \geqslant 2$. We then prove that for every $\alpha > p - 1$, the Keller - Osserman property in $(\mathbb{R}^d, \mathcal{H}_\alpha)$ is valid ; i . e . , every open ball admits a regular Evans function , which yields the validity of the Brelot convergence property . Among others , we prove for $\alpha > p - 1$ a theorem of the Liouville type in the form $\mathcal{H}_\alpha(\mathbb{R}^d) = \{0\}$. Finally in the ninth section , we consider some applications of the previous results to the case of the $p-$ Laplace operator , where we also prove the uniqueness of the regular Evans function for star domain and strict positive b and $\mathcal{H}_\alpha$ for $\alpha > p - 1$.

Note that our methods are applicable to broader class of weighted equations (see [1 0]) . The use of the constant weight $\equiv 1$ is only for sake of simplicity .

2 . Notation

We introduce the basic notation which will be observed throughout this paper . $\mathbb{R}^d$ is the real Euclidean $d-$ space , $d \geq 2$. For an open set U of $\mathbb{R}^d$ and an positive integer $k, \mathcal{C}^k(U)$ is the set of all k t imes continuously differentiable functions on an open set U. $\mathcal{C}^\infty(U) := \bigcap_{k \geq 1} \mathcal{C}^k(U)$ and $\mathcal{C}^\infty_c(U)$ the set of all functions in $\mathcal{C}^\infty(U)$ compactly supported by U. For a measurable set $X, \mathcal{B}(X)$ denotes the set of all Borel numerical functions on X and for $q \geq 1, L^q(X)$ is the $q^{th}-$ power Lebesgue space defined on X. Given any set $\mathcal{Y}$ of functions $\mathcal{Y}_b(\mathcal{Y}^+$ resp .) denote the set of all functions in $\mathcal{Y}$ which are bounded (positive resp .).$\mathcal{W}^{1,q}(U)$ is the $(1, q)-$ Sobolev space on U. $\mathcal{W}^{1,q}_0(U)$ the closure of $\mathcal{C}^\infty_c(U)$ in $\mathcal{W}^{1,q}(U)$, relatively to its norm .

$\mathcal{W}^{-1q',}(U)$ is the dual of $\mathcal{W}_0^{1,q}(U)$, $\quad q' = q(q-1)^{-1}$. $\quad u \wedge v($ resp . $\quad u \vee v)$ is the infinimum (resp . the maximum) of u and $v; u^+ = u \vee 0$ and $u^- = u \wedge 0$.

3 . Existence and Uniqueness of Solutions

Let Ω be a bounded open subset of $\mathbb{R}^d (d \geqslant 1)$. We will investigate the existence of solutions $u \in \mathcal{W}^{1,p}(\Omega), 1 < p \leqslant d$, of the variational Dirichlet problem associated with the quasilinear elliptic equation

$$-\mathrm{div}(\mathcal{A}(x, \nabla u)) + \mathcal{B}(x, u) = 0.$$

In this paper we suppose that the functions $\mathcal{A} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are given Carath é odory functions and the following structure conditions are

satisfied :

(I) $\zeta \to \mathcal{B}(x, \zeta)$ is increasing and $\mathcal{B}(x, 0) = 0$ for every $x \in \mathbb{R}^d$. **(A 1)** There exists $0 < \varepsilon < 1$ such that for any $u \in L^\infty(\mathbb{R}^d)$,

$$\mathcal{B}(., u(.)) \in L^{p-\varepsilon \frac{\text{loc}}{d}}(\mathbb{R}^d).$$

(A 2) There exists $\nu > 0$ such that for every $\xi \in \mathbb{R}^d$,

$$\mid \mathcal{A}(x, \xi) \mid \leqslant \nu \mid \xi \mid^{p-1} .$$

(A 3) There exists $\mu > 0$ such that for every $\xi \in \mathbb{R}^d$,

$$\mathcal{A}(x, \xi).\xi \geqslant \mu \mid \xi \mid^p .$$

(M) For all $\xi, \xi' \in \mathbb{R}^d$ with $\xi \neq \xi'$,

$$[\mathcal{A}(x, \xi) - \mathcal{A}(x, \xi')] \cdot (\xi - \xi') > 0.$$

We recall that assumptions (A 2) , (A 3) and (M) are satisfied in the framework of [1 0] when the admissible weight is $\omega \equiv 1$.

Recall that $u \in \mathcal{W}^{1,p}_{\text{loc}}(\Omega)$ is a *s o lutio n* of (1 . 1) in Ω provided that for all $\phi \in$

$$\mathcal{W}_0^{1,p}(\Omega) \text{and} \mathcal{B}(., u) \in L^{p^{*'}}_{\text{loc}}(\Omega),$$

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \phi dx + \int_\Omega \mathcal{B}(x, u) \phi dx = 0. \quad (3.1)$$

A function $u \in \mathcal{W}^{1,p}_{\text{loc}}(\Omega)$ is termed *subsolutions* (resp . *supersolutions*) of (1 . 1) if for all non - negative functions $\phi \in \mathcal{W}_0^{1,p}(\Omega)$ and $\mathcal{B}(., u) \in L^{p^{*'}}_{\text{loc}}(\Omega)$,

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \phi dx + \int_\Omega \mathcal{B}(x, u) \phi dx \leqslant 0 \quad (\text{resp.} \quad \geqslant 0).$$

If u is a bounded subsolution (resp . bounded supersolution) , then for every $k \geqslant 0$, $u - k($ resp $.u + k)$ is also subsolution (resp . supersolution) for (1 . 1) .

For a positive constant M and $u \in L^p(\Omega)$, we define the truncated function

$$\tau_M(u)(x) = braceleftmid - braceleftbt \begin{matrix} -M & u(x) \leqslant -M \\ u(x) & -M < u(x) < M \\ M, & M \leqslant u(x) \end{matrix}$$

(a . e $.x \in \Omega$). It is clear that the truncation mapping τ_M is bounded and continuous from $L^p(\Omega)$ to itself .

 For $u \in \mathcal{W}^{1,p}(\Omega)$ and $\mathcal{B}(x, \tau_M(u)) \in L^{p^{\rm loc}}_{*'}(\Omega)$, we define $\mathcal{L}_M : \mathcal{W}^{1,p}(\Omega) \to$

$'$

$\mathcal{W}^{-1,p}(\Omega)$as

$$\langle \mathcal{L}_M(u), \phi\rangle := \int_\Omega \mathcal{A}(x, \nabla u)\cdot \nabla\phi dx + \int_\Omega \mathcal{B}(x, \tau_M(u))\phi dx, \quad \phi \in \mathcal{W}^{1,p}_0(\Omega)$$

$'$

here $\langle .,.\rangle$ is the pairing between $\mathcal{W}^{-1,p}(\Omega)$ and $\mathcal{W}^{1,p}(\Omega)$. It follows from Assump - tions (A 1) , (A 2) , (A 3) , and the carath é odory conditions that $\mathcal{L}_M$ is well defined . We consider the variational inequality

$$\langle \mathcal{L}_M(u), v - u\rangle \geqslant 0, \quad \forall v \in \mathcal{K}, u \in \mathcal{K}, \tag{3.2}$$

where $\mathcal{K}$ is a given closed convex set in $\mathcal{W}^{1,p}$(⋉⋊≈⋖∼∼⋊⋋∼⋖⋉≈॥⋖) such that for given $f \in \mathcal{W}^{1,p}$(⋉⋊≈⋖∼∼⋊⋋∼⋖⋉≈॥⋖),

$$\mathcal{K} \subset f + \mathcal{W}^{1,p}_0(⋉⋊≈⋖∼∼⋊⋋∼⋖⋉≈॥⋖).$$

Typical examples of closed convex sets $\mathcal{K}$ are as follows : for $f \in \mathcal{W}^{1,p}$(⋉⋊≈⋖∼∼⋊⋋∼⋖⋉≈॥⋖) and $\psi1, \psi2 : \Omega \to [-\infty, +\infty]$ let the convex set is
$\mathcal{K}^f_{\psi1,\psi2} = \mathcal{K}^f_{\psi1,\psi2}(\Omega) = \{u \in \mathcal{W}^{1,p}(\Omega) : \psi1 \leq u \leqslant \psi2$ a . e . in $\Omega, u - f \in \mathcal{W}^{1,p}_0(\Omega)\}$.

(3 . 3) We write $\mathcal{K}^f_{\psi1} = \mathcal{K}^f_{\psi1,+\infty}(\Omega)$ and , if $f = \psi1 \in \mathcal{W}^{1,p}(\Omega), \mathcal{K}_f = \mathcal{K}^f_{f}.$ A function u satisfying (3 . 2) with $M = +\infty$ and the closed convex sets $\mathcal{K}^f_{\psi1}$ is called a *s o lution to the o bstacle pro b lem* in $\mathcal{K}^f_{\psi1}$. For the notion of obstacle problem , the reader is referred to monograph [1 0 , p . 60] or [1 8 , Chap . 5] . We observe that any solution of the obstacle problem in $\mathcal{K}^f_{\psi1}(\Omega)$ is always a supersolution of the equation (1 . 1) in Ω. Conversely , a supersolution u is always a solution to the obstacle problem in $\mathcal{K}^u_u(\omega)$ for all open $\omega \subset \omega \subset \Omega$. Furthermore a solution u to equation (1 . 1) in an open set Ω is a solution to the obstacle problem in $\mathcal{K}^u_{-\infty}(\omega)$ for all open $\omega \subset \omega \subset \Omega$. Similarly , a solution to the obstacle problem in $\mathcal{K}^u_{-\infty}(\Omega)$ is a solution to (1 . 1) .

For the uniqueness of a solution to the obstacle problem we have following lemma [1 0 , Lemma 3 . 22] :

Lemma 3 . 1 . *Suppose that* u *is a s o lution to the o bstacle problem in* $\mathcal{K}^f_g(\Omega)$. *If* $v \in \mathcal{W}^{1,p}(\Omega)$ *is a supersolution of (1 . 1) in* Ω *such that* $u \wedge v \in \mathcal{K}^f_g(\Omega)$, *th en a . e .*

$$u \leqslant v in \Omega.$$

Theorem 3 . 1 . *Let* $\psi1$ *and* $\psi2$ *in* L^∞(⋉⋊≈⋖∼∼⋊⋋∼⋖⋉≈॥⋖), $f \in \mathcal{W}^{1,p}$(⋉⋊≈⋖∼∼⋊⋋∼⋖⋉≈॥⋖) *and* $\mathcal{K}^f_{\psi1,\psi2}$ *as a bo ve assume that* $\mathcal{K}^f_{\psi1,\psi2}$ *is non empty . Then for every positive constant* M, $\| \psi1 \|_\infty \vee \| \psi2 \|_\infty \leqslant M < +\infty$ *th e variational inequality (3 . 2) has a unique s o lutio n . Moreover , if* $w \in \mathcal{W}^{1,p}(\Omega)$ *is a supersolution (resp . subsolution) to th e equation (1 . 1) such that* $w \wedge u($ *resp* $.w \vee u) \in \mathcal{K}^f_{\psi1,\psi2}$, *then* $u \leqslant w($ *resp* $.w \leqslant u)$.

Proof . Let $\| \psi1 \| \infty \vee \| \psi2 \| \infty \leqslant M < +\infty$. If $u, v \in \mathcal{K}^f_{\psi1,\psi2}$ are solutions of (3 . 2) , it follows from (**I**) and (**M**) that

$$
\begin{aligned}
0 \quad \geqslant \quad & \int_\Omega [\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v)] \cdot \nabla(v-u)dx \\
& + \int_\Omega [\mathcal{B}(x, \tau_M(u)) - \mathcal{B}(x, \tau_M(v))](v-u)dx \\
= \quad & \langle \mathcal{L}_M(u) - \mathcal{L}_M(v), v-u \rangle \geqslant 0,
\end{aligned}
$$

then $v-u$ is constant on connected components of Ω. This , on the other hand , since $v-u\in\mathcal{W}_0^{1,p}(\Omega)$, implies that $v=u$.
To prove the existence we will use [1 2 , Corollary III . 1 . 8 , p . 87] . Since $\mathcal{K}^f_{\psi 1,\psi 2}$ is a non empty closed convex subset of $\mathcal{W}^{1,p}(\Omega)$, it is enough to prove that $\mathcal{L}_M$ is monotone , coercive and weakly continuous on $\mathcal{K}^f_{\psi 1,\psi 2}$. We have

$$\begin{aligned}\langle\mathcal{L}_M(u)-\mathcal{L}_M(v),u-v\rangle &= \int_\Omega[\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla v)]\cdot\nabla(u-v)dx+\\ &\quad+\int_\Omega[\mathcal{B}(x,\tau_M(u))-\mathcal{B}(x,\tau_M(v))].(u-v)dx\end{aligned}$$

for all $v,u\in\mathcal{K}^f_{\psi 1,\psi 2}$ and the structure conditions on $\mathcal{A}$ and $\mathcal{B}$ yield that $\mathcal{L}_M$ is monotone and coercive (for the definition of monotone or coercive operator the reader is referred to [14 , 1 2]) .
To show that $\mathcal{L}_M$ is weakly continuous on $\mathcal{K}^f_{\psi 1,\psi 2}$, let $(u_n)_n\subset\mathcal{K}^f_{\psi 1,\psi 2}$ be a se - quence that converges to $u\in\mathcal{K}^f_{\psi 1,\psi 2}$. There is a subsequence $(u_{n_k})k$ such that $u_{n_k}\to u$ and $\nabla u_{n_k}\to\nabla u$ pointwise a . e . in Ω. Since $\mathcal{A}$ and $\mathcal{B}$ are Carath é odory functions , $\mathcal{A}(.,\nabla u_{n_k})$ and $\mathcal{B}(.,\tau_M(u_{n_k}))$ converges in measure to $\mathcal{A}(.,\nabla u)$ and $\mathcal{B}(x,\tau_M(u))$ respectively [1 1] . Pick a subsequence , indexed also by n_k, such that $\mathcal{A}(.,\nabla u_{n_k})$ and $\mathcal{B}(.,\tau_M(u_{n_k}))$ converges pointwise a . e . in Ω to $\mathcal{A}(.,\nabla u)$ and $\mathcal{B}(x,\tau_M(u))$ respectively . Because $(u_{n_k})_{n_k}$ is bounded in $\mathcal{W}^{1,p}(\Omega)$, it follow that $(\mathcal{A}(.,\nabla u_{n_k}))_k$ is bounded in $(L^p_{p-}1(\Omega))d$ and that $\mathcal{A}(.,\nabla u_{n_k})\rightharpoonup\mathcal{A}(.,\nabla u)$ weakly in $(L^p_{p-}1(\Omega))^d_.$ We have also $\mathcal{B}(.,\tau_M(u_{n_k}))\rightharpoonup\mathcal{B}(.,\tau_M(u))$ weakly in $L^{p^{*'}}(\Omega)$. Since the weak limits are independent of the choice of the subsequence , we have for all $\phi\in\mathcal{W}_0^{1,p}(\Omega)$

$$\langle\mathcal{L}_M(u_n),\phi\rangle\to\langle\mathcal{L}_M(u),\phi\rangle$$

and hence $\mathcal{L}_M$ is weakly continuous on $\mathcal{K}^f_{\psi 1,\psi 2}$.
Let now $w\in\mathcal{W}^{1,p}(\Omega)$ be a supersolution of the equation (1 . 1) such that $u\wedge w\in\mathcal{K}^f_{\psi 1,\psi 2}$, then $u-(u\wedge w)\in\mathcal{W}_0^{1,p}(\Omega)$ and we have

$$\begin{aligned}0 &\leqslant \int_\Omega[\mathcal{A}(x,\nabla w)-\mathcal{A}(x,\nabla u)]\cdot\nabla(u-(u\wedge w))dx+\\ &\quad+\int_\Omega[\mathcal{B}(x,\tau_M(w))-\mathcal{B}(x,\tau_M(u))].(u-(u\wedge w))dx\\ &= \int_{\{u>w\}}[\mathcal{A}(x,\nabla(u\wedge w))-\mathcal{A}(x,\nabla u)]\cdot\nabla(u-(u\wedge w))dx+\\ &\quad+\int_{\{u>w\}}[\mathcal{B}(x,\tau_M(u\wedge w))-\mathcal{B}(x,\tau_M(u))].(u-(u\wedge w))dx\\ &\leqslant 0.\end{aligned}$$

It follow , by (**I**) and (**M**) , that $\nabla(u-(u\wedge w))=0$ a . e . in Ω and hence $u\leqslant w$ a . e . in Ω. The same proof is valid if w is a subsolution . □
As an application of Theorem 3 . 1 , we have the following two theorems . **Theorem 3 . 2 .** *Let* $f\in\mathcal{W}^{1,p}(⋉⋊≈⋖∼∼⋊⋋∼⋖⋉≈⫴⋖)\cap L^\infty(⋉⋊≈⋖∼∼⋊⋋∼⋖⋉≈⫴⋖)$ *and*

$$\mathcal{K}=\{u\in\mathcal{W}^{1,p}(\Omega):f\le u\leqslant\ \|f\|_\infty\quad a.\quad e.,u-f\in\mathcal{W}_0^{1,p}(\Omega)\}.$$

Then there exists $u \in \mathcal{K}$ *such that*

$$\langle \mathcal{L}(u), v-u\rangle \geqslant 0 \quad for\ al\ l\ \ v \in \mathcal{K}.$$

Moreover ,u is a supersolution of (1 . 1) in Ω. *Proof .* For $m>0$, by Theorem 3 . 1 there exists a unique function u_m in

$$\mathcal{K}^f_{f,\|}f\ \|_\infty +m = \{u \in \mathcal{W}^{1,p}(\Omega) : f \leqslant u \leqslant \ \| f \| \infty + m \text{a.e.}, u-f \in \mathcal{W}^{1,p}_0(\Omega)\}$$

such that

$$\langle \mathcal{L}_{\|}f\ \|_\infty +m(u_m), v-u_m\rangle \geqslant 0$$

for all $v \in \mathcal{K}^f_{f,\|}f\ \|_\infty +m\cdot$ Since $u_m- \ \| f \| \infty = u_m - f + f - \| f \|_\infty \ \leqslant u_m - f$ and $(u_m-f)^+ \geqslant (u_m - \| f \|_\infty)^+$, we have $\eta := (u_m - \| f \|_\infty)^+ \in \mathcal{W}^{1,p}_0(\Omega)($ see e . g . [1 0 , Lemma 1 . 25]) . Moreover , since $u_m - \eta \in \mathcal{K}^f_{f,\|}f\ \|_\infty +m$ and $\| f \|_\infty$ is a supersolution of (1 . 1) , we have

$$0 \quad \leqslant \quad -\int_\Omega \mathcal{A}(x,\nabla u_m)\cdot \nabla \eta dx - \int_\Omega [\mathcal{B}(x,u_m) - \mathcal{B}(x, \| f \|_\infty)]\eta dx$$

$$= \quad -\int_{\{u_m>} \ \| f \| \infty\} \mathcal{A}(x,\nabla u_m)\cdot \nabla u_m dx +$$

$$-\int_{\{u_m>\|} f\ \|_\infty\} [\mathcal{B}(x,u_m) - \mathcal{B}(x, \| f \| \infty^{)]}(u_m - \| f \|_\infty) dx$$

$$\leqslant \quad 0,$$

then $\nabla\eta = 0$ a . e . in Ω by (**M**) . Because $\eta \in \mathcal{W}^{1,p}_0(\Omega), \eta = 0$ a . e . in Ω. It follows that $u_m \leqslant \ \| f \| \infty$ a . e . in Ω. It follows that $u_m \leqslant \ \| f \|_\infty$ a . e . in Ω, and therefore $f \ \leqslant u_m \ < \ \| f \| \infty + m$ a . e . in Ω. Given a non - negative $\phi \in \mathcal{C}^\infty_c(\Omega)$ and $\varepsilon > 0$ sufficiently small such that $u_m + \varepsilon\phi \in \mathcal{K}^f_{f,\|}f\ \|_\infty +m$, consequently

$$\langle \mathcal{L}(u_m), \phi\rangle \geqslant 0$$

which means that u_m is a supersolution of (1 . 1) in Ω. □ **Theorem 3 . 3 .** *Let* Ω *be a bounded open s e t of* $\mathbb{R}^d, f \in \mathcal{W}^{1,p}(\Omega) \cap L^\infty(\Omega)$. *Then there is a unique function* $u \in \mathcal{W}^{1,p}(\Omega)$ *with* $u - f \in \mathcal{W}^{1,p}_0(\Omega)$ *such that*

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\phi dx + \int_\Omega \mathcal{B}(x,u)\phi dx = 0,$$

$$whenever \phi \in \mathcal{W}^{1,p}_0(\Omega).$$

Proof . For $m>0$, by Theorem 3 . 1 , there exists a unique u_m in

$$\mathcal{K}_{f,m} := \{u \in \mathcal{W}^{1,p}(\Omega) : | u | \leqslant \ \| f \| \infty + m \text{a.e.}, u - f \in \mathcal{W}^{1,p}_0(\Omega)\},$$

such that

$$\langle \mathcal{L}_{\|}f\ \|_\infty +m(u_m), v-u_m\rangle \geqslant 0,$$

for all $v \in \mathcal{K}_{f,m}$. Since $u_m + \| f \|_\infty = u_m - f + f + \| f \|_\infty \geqslant u_m - f$ and $(u_m-f)^- \leqslant (u_m + \ \| f \| \infty^) \wedge 0$, we have $\eta := (u_m + \| f \|_\infty) \wedge 0 \in \mathcal{W}^{1,p}_0(\Omega)$ (see

e . g . [1 0 , Lemma 1 . 25]) . Moreover , since $\eta + u_m \in \mathcal{K}_{f,m}$ and $- \parallel f \parallel \infty$ is a subsolu -
tion of (1 . 1) , we have

$$0 \leqslant \int_\Omega \mathcal{A}(x, \nabla u_m) \cdot \nabla \eta dx + \int_\Omega [\mathcal{B}(x, u_m) - \mathcal{B}(x, - \parallel f \parallel_\infty)]\eta dx$$

$$= - \int_{\{u_m < - \parallel f \parallel \infty\}} \mathcal{A}(x, \nabla u_m) \cdot \nabla u_m dx +$$

$$- \int_{\{u_m < - \parallel f \parallel_\infty\}} [\mathcal{B}(x, u_m) - \mathcal{B}(x, - \parallel f \parallel_\infty)](u_m + \parallel f \parallel_\infty) dx$$

$$\leqslant 0,$$

then $\nabla \eta = 0$ a . e . in Ω by (**M**) . Because $\eta \in \mathcal{W}_0^{1,p}(\Omega), \eta = 0$ a . e . in Ω. It follows that $- \parallel f \parallel \infty \leqslant u_m$ a . e . in Ω. Note that $-u_m$ is also a solution in $\mathcal{K}_{-f,m}$ of the following variational inequality

$$\langle \widetilde{L}_{\parallel f \parallel \infty + m(u)}, v - u \rangle = \int_\Omega \widetilde{A}(x, \nabla u) \cdot \nabla (v - u) dx$$

$$+ \int_\Omega \widetilde{B}(x, \tau_{\parallel f \parallel \infty + m(u)})(v - u) dx \geqslant 0,$$

where $\widetilde{A}(., \xi) = -\mathcal{A}(., -\xi)$ and $\widetilde{B}(., \zeta) = -\mathcal{B}(., -\zeta)$ which satisfy the same as - sumptions as $\mathcal{A}$ and $\mathcal{B}$. It follows that $u_m \leqslant \parallel f \parallel_\infty$ a . e . in Ω, and therefore
$| u_m| < \parallel f \parallel \infty + m$ a . e . in Ω. Given $\phi \in \mathcal{C}_c^\infty(\Omega)$ and $\varepsilon > 0$ sufficiently small such

that $u_m \pm \varepsilon \phi \in \mathcal{K}_{f,m}$, consequently

$$\langle \mathcal{L}(u_m), \phi \rangle = 0$$

which means that u_m is a desired function . □

By regularity theory (e . g . [1 8 , Corollary 4 . 1 0]) , any bounded solution of (1 . 1) can be redefined in a set of measure zero so that it becomes continuous .

Definition 3 . 1 . A relatively compact open set U is called $p-$*regularity* if , for each function $f \in \mathcal{W}^{1,p}(U) \cap \mathcal{C}(U)$, the continuous solution u of (1 . 1) in U with $u - f \in \mathcal{W}^{1,p}(U)$ satisfies $\lim_{x \to y} u(x) = f(y)$ for all $y \in \partial U$.

A relatively compact open set U is called regular , if for every continuous function f on ∂U, there exists a unique continuous solution u of (1 . 1) on U such that

$$\lim_{x \to y} u(x) = f(y) \text{for all} y \in \partial U.$$

If U is $p-$*hyphen* regular and $f \in \mathcal{W}^{1,p}(U) \cap \mathcal{C}(U)$, then the solution u given by Theo - rem 3 . 3 satisfies

$$\lim_{\in_x U, x \to z} u(x) = f(z)$$

for all $z \in \partial U$[18, Corollary 4 . 1 8] .

4 . Comparison Principle and Dirichlet Problem

The following *comparison principle* is useful for the potential theory associated with equation (1 . 1) : **Lemma 4 . 1 .** *Suppose that* u *is a supersolution and* v *is a subsolution on* Ω *such that*

$$\limsup_{x \to y} v(x) \leqslant \liminf_{x \to y} u(x)$$

 for al l $y \in \partial\Omega$ *and if both s ides of the inequality are not s imultaneously* $+\infty$ *o r*

$$-\infty, then v \leqslant u in \Omega.$$

Proof . By the regularity theory (see e . g . [1 8 , Corollary 4 . 1 0]) , we may assume that u is lower semicontinuous and v is upper semicontinuous on Ω. For fixed $\varepsilon > 0$, the set $K_\varepsilon = \{x \in \Omega : v(x) \geqslant u(x) + \varepsilon\}$ is a compact subset of Ω and therefore $\phi = (v - u - \varepsilon)^+ \in \mathcal{W}_0^{1,p}(\mathbb{R}^d)$. Testing by ϕ, we obtain

$$\begin{aligned} &\int_{\{v>u+\varepsilon\}} [\mathcal{A}(x, \nabla(u+\varepsilon)) - \mathcal{A}(x, \nabla v)] \cdot \nabla\phi dx \\ &\quad + \int_{\{v>u+\varepsilon\}} [\mathcal{B}(x, u+\varepsilon) - \mathcal{B}(x, v)]\phi dx \geqslant 0 \end{aligned} \tag{4.1}$$

Using Assumptions (I) and (M) we have

$$\int_{\{v>u+\varepsilon\}} [\mathcal{A}(x, \nabla u + \varepsilon) - \mathcal{A}(x, \nabla v)] \cdot \nabla(v - u - \varepsilon) dx = 0$$

and again by M we infer that $v \leqslant u + \varepsilon$ on Ω. Letting $\varepsilon \to 0$ we have $v \leqslant u$ on

$$\Omega. \quad \square$$

Theorem 4 . 1 . *Every p− regular s e t is regular in the s ense of definition 3 . 1 .*

Proof . Let Ω be a $p-$ regular set in $\mathbb{R}^d$ and f be a continuous function on $\partial\Omega$. We shall prove that there exists a unique continuous solution u of (1 . 1) on Ω such that $\lim_{x\to y} u(x) = f(y)$ for all $y \in \partial\Omega$. The uniqueness is given by Lemma 4 . 1 . By [1 8 , Theorem 4 . 1 1] we have the continuity of u. For the existence , we may suppose that $f \in \mathcal{C}_c(\mathbb{R}^d)$(Tietze ' s extension theorem) . Let fi be a sequence of functions from $\mathcal{C}_c^1(\mathbb{R}^d)$ such that $| fi - f | \leqslant 2^{-i}$ and $| fi | + | f | \leqslant M$ on Ω for the same constant M and for all i. Let $u_i \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\Omega)$ be the unique solution for the Dirichlet problem with boundary data fi(Theorem 3 . 3) . Then from Lemma 4 . 1 we deduce that $| u_i - u_j | \leqslant 2^{-i} + 2^{-j}$ and $| u_i | \leqslant M$ on Ω for all i and j. We denote by u the limit of the sequence $(u_i)i$. We will show that u is a lo cal solution of the equation . For this , we prove that the sequence $(\nabla u_i)i$ is lo cally uniformly bounded in $(L^p(\Omega))^d$. Let $\phi = -\eta^p u_i, \eta \in \mathcal{C}_c^\infty(\Omega), 0 \leqslant \eta \leqslant 1$ and $\eta = 1$ on $\omega \subset \omega \subset \Omega$.
Since $\phi \in \mathcal{W}_0^{1,p}(\Omega)$, we have

$$\begin{aligned} 0 &= \int_\Omega \mathcal{A}(x, \nabla u_i) \cdot \nabla\phi dx + \int_\Omega \mathcal{B}(x, u_i)\phi dx \\ &= \int_\Omega \mathcal{A}(x, \nabla u_i).(-\eta^p \nabla u_i - pui\eta^{p-1}\nabla\eta) dx - \int_\Omega \eta^p \mathcal{B}(x, u_i) u_i dx \\ &\leqslant -\mu \int_\Omega \eta^p | \nabla u_i |^p dx + p\nu \int_\Omega \eta^{p-1} | \nabla u_i |^{p-1} | u_i || \nabla\eta | dx + C(M, \| \eta \|_\infty, | \Omega |), \end{aligned}$$

and therefore , using the Young inequality , we obtain

$$\begin{aligned} &\int_\Omega \eta^p | \nabla u_i |^p dx \\ &\leqslant p^{\varepsilon^p} \nu_\prime^\mu \int_\Omega \eta^p | \nabla u_i |^p dx + \nu p_{\varepsilon^p \mu} \int_\Omega | u_i |^p | \nabla\eta |^p dx + C(M, \| \eta \|_\infty, | \Omega |) \\ &\leqslant \nu^\mu_{p^{\varepsilon^p}\prime} \int_\Omega \eta^p | \nabla u_i |^p dx + C(M, \| \eta \|_\infty, | \Omega |, \| \nabla\eta \|_\infty, \varepsilon). \end{aligned}$$

$$\text{If} 0 < \varepsilon < \left(\begin{array}{c} c_1 \\ pa1 \end{array} \right) p - p1, \text{then}$$

$$\int_{\omega} | \nabla u_i |^p \, dx \leqslant \mu C(M, \| \eta_{\mu} \|_{\infty -} r^{\nu}_{,p}{}_{|\Omega|}{}_{\varepsilon^p}{}_{, \|} \nabla \eta \|_{\infty}, \varepsilon) \text{forall} i.$$

It follows that the sequence $(u_i)i$ is lo cally uniformly bounded in $\mathcal{W}^{1,p}(\Omega)$. Fix $D \Subset G \Subset \Omega$. Since $(u_i)i$ converges pointwise to u and by [1 0 , Theorem 1 . 32] , we obtain that $u \in \mathcal{W}^{1,p}(D)$ and $(u_i)i$ converges weakly , in $\mathcal{W}^{1,p}(D)$, to u. Let $\eta \in \mathcal{C}_0^{\infty}(G)$ such that $0 \leqslant \eta \leqslant 1, \eta = 1$ in D and testing by $\phi = \eta(u - u_i)$ for the solution u_i, we have

$$\begin{aligned} & -\int_G \eta \mathcal{A}(x, \nabla u_i) \cdot \nabla (u - u_i) dx \\ = & \int_G (u - u_i) \mathcal{A}(x, \nabla u_i) \cdot \nabla \eta dx + \int_G \eta \mathcal{B}(x, u_i)(u - u_i) dx \\ \leqslant & (\int_G | u - u_i |^p \, dx) 1/p [C + \nu (\int_G | \nabla u_i |^p \, dx) p - p1] \\ \leqslant & C (\int_G | u - u_i |^p \, dx) 1/p. \end{aligned}$$

Since

$$\begin{aligned} 0 \leqslant & \int_D [\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_i)] \cdot \nabla (u - u_i) dx \\ \leqslant & \int_G \eta \mathcal{A}(x, \nabla u) \cdot \nabla (u - u_i) dx + C (\int_G | u - u_i |^p \, dx) 1/p \end{aligned}$$

and the weak convergence of $(\nabla u_i)i$ to ∇u implies that

$$\lim_{i \to \infty} \int_G \eta \mathcal{A}(x, \nabla u) \cdot \nabla (u - u_i) dx = 0,$$

we conclude

$$\lim_{i \to \infty} \int_D [\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_i)] \cdot \nabla (u - u_i) dx = 0.$$

Now [1 0 , Lemma 3 . 73] implies that $\mathcal{A}(x, \nabla u_i)$ converges to $\mathcal{A}(x, \nabla u)$ weakly in

$$(L^{p'}(D))^n_.$$

Let $\psi \in \mathcal{C}_0^{\infty}(G)$. By the continuity in measure of the Carath é odory function $\mathcal{B}(x, z)$ [11] and by using the domination convergence theorem (in measure) , we have

$$\lim_{i \to \infty} \int_{\Omega} \mathcal{B}(x, u_i) \psi dx = \int_{\Omega} \mathcal{B}(x, u) \psi dx.$$

Finally we obtain

$$\begin{aligned} 0 = & \lim_{i \to \infty} [\textstyle\int_{\Omega} \mathcal{A}(x, \nabla u_i) \cdot \nabla \psi dx + \displaystyle\int_{\Omega} \mathcal{B}(x, u_i) \psi dx] \\ = & \int_{\Omega} \mathcal{A}(x, \nabla u) \cdot \nabla \psi dx + \int_{\Omega} \mathcal{B}(x, u) \psi dx. \end{aligned}$$

By an application of [1 8 , Corollay 4 . 1 8] for each u_i we obtain

$$x \in \lim_{\Omega, x \to z} u_i(x) = fi(z)$$

 for all $z \in \partial\Omega$. From the following estimation , of u on all Ω,

$$u_i - 2^{-i} \leqslant u \leqslant u_i + 2^{-i} \text{for all} i$$

we deduce that for all i

$$fi(z) - 2^{-i} \leqslant x \to z^{x\in\Omega}_{\liminf} u(z) \leqslant x \to z^{x\in\Omega}_{\limsup} u(z) \leqslant fi(z) + 2^{-i}.$$

Letting $i \to \infty$ we obtain

$$\lim_{x\to z} u(x) = f(z)$$

for all $z \in \partial\Omega$ which finishes the proof . □

Corollary 4 . 1 . *There exists a basis $\mathcal{V}$ of regular s e ts which is s table by inters ection i . e . for e very U and V in $\mathcal{V}$, we have $U \cap V \in \mathcal{V}$.*

The proof of this corollary can be found in Theorem 4 . 1 and [1 0 , Corollary 6 . 32] .

For every open set V and for every $f \in \mathcal{C}(\partial V)$ we shall denote by $H_V f$ the *s o lutio n of th e Dirichlet problem* for the equation (1 . 1) on V with the boundary data f.

5 . Nonlinear Potential Theory associated with the equation (1 . 1)

For every open set U we shall denote by $\mathcal{U}(U)$ the set of all relatively compact open , regular subset V in U with $V \subset U$.

By previous section and in order to obtain an axiomatic nonlinear potential theory , we shall investigate the harmonic sheaf associated with (1 . 1) and defined as follows : For every open subset U of $\mathbb{R}^d (d \geqslant 1)$, we set

$$\begin{aligned}\mathcal{H}(U) &= \{u \in \mathcal{C}(U) \cap \mathcal{W}^{1,p}_{\text{loc}}(U) : u \text{ is a solution of (1.1)}\} \\ &= \{u \in \mathcal{C}(U) : H_V u = u \text{for every} V \in \mathcal{U}(U)\}.\end{aligned}$$

Element in the set $\mathcal{H}(U)$ are called *harmonic* on U.

We recall (see [4]) that $(X, \mathcal{H})$ satisfies the *Bauer convergence property* if for every subset U of X and every monotone sequence $(h_n)_n$ in $\mathcal{H}(U)$, we have $h = \lim_{n\to\infty} h_n \in \mathcal{H}(U)$ if it is lo cally bounded .

Proposition 5 . 1 . *Let be U an open subset of $\mathbb{R}^d$. Then every family $\mathcal{F} \subset \mathcal{H}(U)$ of locally uniformly bounded harmonic functions is equicontinuous .*

Proof . Let $V \subset V \subset U$ and a family $\mathcal{F} \subset \mathcal{H}(U)$ of lo cally uniformly bounded harmonic functions . Then sup $\{| u(x) | : x \in V$ and $u \in \mathcal{F}\} < \infty$ and by [1 8] , is equicontinuous on V. □

Corollary 5 . 1 . *We have the Bauer convergence properties and moreover every locally bounded family of harmonic functions on an open s e t is relatively compact .*

Proof . Let U be an open set and $\mathcal{F}$ a lo cally bounded subfamily of $\mathcal{H}(U)$. By Proposition 5 . 1 , there exist a sequence $(u_n)_n$ in $\mathcal{F}$ which converge to u on U lo cally uniformly . Let now $V \in \mathcal{U}(U)$. For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $u - \varepsilon \leqslant u_n \leqslant u + \varepsilon$ for every $n \geqslant n_0$. The comparison principle yields therefore $(H_V u) - \varepsilon \leqslant u_n \leqslant (H_V u) + \varepsilon$, thus $(H_V u) - \varepsilon \leqslant u \leqslant (H_V u) + \varepsilon$. Letting $\varepsilon \to 0$, we

$$\text{get} u = H_V u. \quad \square$$

Proposition 5 . 2 . [4] *Let V a regular subset of $\mathbb{R}^d$ and le t $(f_n)_n$ and f in $\mathcal{C}(\partial V)$ such that $(f_n)_n$ is a monotone s equence converging to f. Then $\sup_n H_V f_n$ converge*

$$toH_V f.$$

Proof . Let V a regular subset of $\mathbb{R}^d$ and let $(f_n)_n$ and f in $\mathcal{C}(\partial V)$ such that $(f_n)_n$ is increasing to f. Then , by Lemma 4 . 1 , we have

$$\sup_n H_V f_n \leqslant H_V f$$

and , by Corollary 5.1 $\sup_n$ H $V f_n \in \mathcal{H}(V)$. Moreover , For every n and every $z \in \partial V$ we have

$$f_n(z) \leq \lim_{x\to z} \inf(\sup_n H_V f_n(x)) \leq \lim_{x\to z} \sup(\sup_n H_V f_n(x)) \leqslant f(z).$$

Letting n tend to infinity we obtain that

$$f(z) = \lim_{x\to z} (\sup_n H_V f_n)(x).$$

By Lemma 4 . 1 , this shows that in fact H $V f = \sup_n H_V f_n$. An analogous proof can be given if $(f_n)_n$ is decreasing .

□

Corollary 5 . 2 . [4] *Let V be a regular subset of $\mathbb{R}^d$ and $(f_n)_n$ and $(gn)_n$ to s equences in $\mathcal{C}(\partial V)$ which are monotone in the same s ense such that $\lim_n f_n = \lim_n gn$. Then*

$$\lim_n H_V f_n = \lim_n H_V gn.$$

Proof . We assume without loss the generality that (f_n) and (gn) are both increasing . Obviously , $H_V(gn \wedge f_m) \leqslant$ H $V gn$ for every n and m in $\mathbb{N}$, hence $\sup_n H_V(gn \wedge f_m) \leqslant \sup_n H_V gn$ for every m. Since the sequence $(gn \wedge f_m)_n$ is increasing to f_m, the previous proposition implies that H $V f_m \leqslant \sup_n H_V gn$. We then have $\sup_n H_V f_n \leqslant$ $\sup_n$ H $V gn$. Permuting (f_n) and (gn) we obtain the converse inequality . □

Let V be a regular subset of $\mathbb{R}^d$. For every lower bounded and lower semicon - tinuous function v on ∂V we define the set

$$H_V v = \sup_n \{H_V f_n : (f_n)_n \text{ in } \mathcal{C}(\partial V) \text{ and increasing to } v\}.$$

For every upper bounded and upper semicontinuous function u on ∂V we define

$$H_V u = \inf_n \{H_V f_n : (f_n)_n \text{ in } \mathcal{C}(\partial V) \text{ and decreasing to } u\}.$$

Let be U an open set of $\mathbb{R}^d$. A lower semicontinuous and lo cally lower bounded function u from U to $\mathbb{R}$ is termed *hyperharmonic* on U if $H_V u \leqslant u$ on V for all V in $\mathcal{U}(U)$. A upper semicontinuous and lo cally upper bounded function v from U to $\mathbb{R}$ is termed *hypoharmonic* on U if $H_V u \geqslant u$ on V for all V in $\mathcal{U}(U)$. We will denote by $*_{\mathcal{H}}(U)$(resp . $*^{\mathcal{H}}(U)$) the set of all hyperharmonic (resp . hypoharmonic) functions on U.

For $u \in asteriskmath - H(U), v \in *^{\mathcal{H}}(U)$ and $k \geqslant 0$ we have $u + k \in *_{\mathcal{H}}(U)$ and $v - k \in *^{\mathcal{H}}(U)$.
Indeed , let $V \in \mathcal{U}(U)$ and a continuous function such that $g \leqslant u + k$ on ∂V, then $H_V(g - k) \leqslant H_V u \leqslant u$. Since $(H_V g) - k \leqslant H_V(g - k)$, we therefore get $H_V g \leqslant u + k$

$$\mathrm{and thus} u + k \in *_{\mathcal{H}}(U).$$

We have the following comparison principle :

Lemma 5 . 1 . *Suppose that u is hyperharmonic and v is hypoharmonic on an open s e t U. If*

$$\lim_{U\ni x\to y}\sup v(x) \leqslant \overset{\inf u(x)}{\lim_{U\ni x\to y}}$$

for all $y\in\partial U$ *and if both s ides of the previous in equality are not s imultaneously* $+\infty$ *o r* $-\infty$, *then* $v\leqslant u$ *in* U.

The proof is the same as in [1 0 , p . 1 33] .

6 . Sheaf Property for Hyperharmonic and Hypoharmonic Functions

For open subsets U of $\mathbb{R}^d$, we denote by $\mathcal{S}(U)$ (resp . by $\mathcal{S}(U)$) the set of all supersolutions (resp . subsolutions) of the equation (1 . 1) on U.

Recall that a map $\mathfrak{F}$ which to each open subset U of $\mathbb{R}^d$ assigns a subset $\mathfrak{F}(U)$ of $\mathfrak{B}(U)$ is called sheaf if we have the following two properties :

(*Presheaf Property*) For every two open subsets U, V of $\mathbb{R}^d$ such that $U \subset V$,

$$\mathfrak{F}(V)_{|}U\subset\mathfrak{F}(U)$$

(*Localization Property*) For any family $(U_i)_{i\in I}$ of open subsets and any numerical function h on $U=\bigcup_{i\in I}U_i, h\in\mathfrak{F}(U)$ if $h_{|U_i} \in\mathfrak{F}(U_i)$ for every $i\in I$.

An easy verification gives that $\mathcal{S}$ and $\mathcal{S}$ are sheaves . Furthermore , we have the following results which generalize many earlier [1 7 , 2 , 7 , 1 0] .

Theorem 6 . 1 . *Let* U *be a non empty open subset in* $\mathbb{R}^d$ *and* $u\in asteriskmath-H(U)\cap\mathfrak{B}_b(U)$. *Then* u *is a supersolution on* U.

Proof . First , we shall prove that for every open $O \subset O \subset U$, there exists an increasing sequence $(u_i)i$ in in O of supersolutions such that $u=\lim_{i\to\infty}u_i$ on O.

Let $(\phi i)i$ be an increasing sequence in $\mathcal{C}_c^\infty(U)$ such that $u=\sup_i\phi i$ on O. Let u_i be the solution of the obstacle problem in the non empty convex set

$$\mathcal{K}_i:=\{v\in\mathcal{W}^{1,p}(O):\phi i\leqslant v\leqslant \;\|\phi i\|\infty+\;\|\phi i+1\|\infty\quad\text{and} v-\phi i\in\mathcal{W}_0^{1,p}(O)\}.$$

The existence and the uniqueness are given respectively by Theorem 3 . 1 ; moreover is a supersolution (Theorem 3 . 2) . Since u_{i+1} is a supersolution and $u_i\wedge u_{i+1}\in\mathcal{K}_i$, we have $u_i\leqslant u_{i+1}$ in O. We have to prove that the sequence $(u_i)i$ is increasing to u. Let x_0 be an element of the open subset $G_i:=\{x\in O:\phi i(x)<u_i(x)\}$ and ω be a domain such that $x_0\in\omega\subset\omega\subset G_i$. Since for every $\psi\in\mathcal{C}_c^\infty(\omega)$ and for sufficiently

$$\text{small}\mid\varepsilon\mid u_i\pm\varepsilon\psi\in\mathcal{K}_i,$$

$$\int_\omega\mathcal{A}(x,\nabla u_i)\cdot\nabla\psi dx+\int_\omega\mathcal{B}(x,u_i)\psi dx=0.$$

Then u_i is a solution of the equation (1 . 1) on ω and by the sheaf property of $\mathcal{H}$, u_i is a solution of the equation (1 . 1) on G_i. Now the comparison principle implies that $u_i\leqslant u$ on G_i, hence $\phi i\leqslant u_i\leqslant u$ on O and therefore $u=\sup_i u_i$. Finally , the boundedness of the sequence $(u_i)i$ and the same techniques in the proof of Theorem 4 . 1 yield that $(u_i)i$ is lo cally bounded in $\mathcal{W}^{1,p}(O)$ and that u is a supersolution of the equation (1 . 1) in O. $\square$

Corollary 6 . 1 . *Let* U *be a non empty open subset in* $\mathbb{R}^d$ *and* $u\in\mathcal{W}_{loc}^{1,p}(U)\cap*_{\mathcal{H}}(U)$. *Then* u *is a supersolution on* U. *Moreover the infinimum of two supersolutions is als o a supersolution .*

 Proof . Let $u \in \mathcal{W}^{1,p}_{\rm loc}(U) \cap *_{\mathcal{H}}(U)$. The Theorem 6 . 1 implies that $u \wedge n$ is a super - solution for all $n \in \mathbb{N}$, consequently we have for every positive $\phi \in \mathcal{C}^\infty_c(U)$

$$\begin{aligned} 0 &\leqslant \int_U \mathcal{A}(x, \nabla(u \wedge n)) \cdot \nabla \phi dx + \int_U \mathcal{B}(x, u \wedge n)\phi dx \\ &= \int_{\{u<n\}} \mathcal{A}(x, \nabla u) \cdot \nabla \phi dx + \int_U \mathcal{B}(x, u \wedge n)\phi dx. \end{aligned}$$

Letting $n \to +\infty$ we obtain

$$0 \leqslant \int_U \mathcal{A}(x, \nabla u) \cdot \nabla \phi dx + \int_U \mathcal{B}(x, u)\phi dx$$

for all positive $\phi \in \mathcal{C}^\infty_c(U)$, thus u is a supersolution . Moreover , if u and v are two supersolutions then $u \wedge v \in \mathcal{W}^{1,p}_{\rm loc}(U) \cap asteriskmath - H(U)$ so $u \wedge v$ is a supersolution . □

Theorem 6.2. *$asteriskmath - H$ is a sheaf .*

Proof . Let $(U_i)i \in I$ be a family of open subsets of $\mathbb{R}^d, U = \bigcup_{i\in I} U_i$ and $h \in asteriskmath - H(U_i)$
for every $i \in I$. Then by the definition of hyperharmonic function , we have $h \wedge n \in asteriskmath - H(U_i)$ for every $(i, n) \in I \times \mathbb{N}$ and by Theorem 6.1, $h \wedge n$ is a supersolution on each U_i. Since $\mathcal{S}$ is a sheaf , we get $h \wedge n \in \mathcal{S}(U) \subset *_{\mathcal{H}}(U)$. Thus $h = \sup_n h \wedge n \in asteriskmath - H(U)$ and $*_{\mathcal{H}}$ is a sheaf . □

Remark 6 . 1 . For every open subset U of $\mathbb{R}^d$, let $\widetilde{H}(U)$ denote the set of all $u \in \mathcal{W}^{1,p}(U) \cap \mathcal{C}(U)$ such that $\widetilde{B}(x, u) \in L^{p{\rm loc}}_{*'}(U)$ and

$$\int_U \mathcal{A}(x, \nabla u) \cdot \nabla \phi dx + \int_U \widetilde{B}(x, u)\phi dx = 0$$

for every $\phi \in \mathcal{W}^{1,p}_0(U)$, where $\widetilde{B}(x,\zeta) = -\widetilde{B}(x,-\zeta)$. It is easy to see that the mapping $\zeta \to \widetilde{B}(x,\zeta)$ is increasing and that $u \in \mathcal{H}(U)$ if and only if $-u \in \widetilde{H}(U)$. Furthermore $\mathcal{H}$ and $\widetilde{H}$ have the same regular sets and for every $V \in \mathcal{U}(U)$ and $f \in \mathcal{C}(\partial V)$ we have $H_V f = -\widetilde{H}V(-f)$. It follows that $u \in *^{\mathcal{H}}(U)$ if and only if $-u \in \widetilde{asteriskmath} - H(U)$ and therefore $*^{\mathcal{H}}$ is a sheaf .

7 . The degeneracy of the sheaf $\mathcal{H}$

As in the previous section we consider the sheaf $\mathcal{H}$ defined by (1 . 1) . Recall that the *Harnack inequality* or the *Harnack principle* is satisfied by $\mathcal{H}$ if for every domain U of $\mathbb{R}^d$ and every compact subset K in U, there exists two constants $c_1 \geqslant 0$ and $c_2 \geqslant 0$ such that for every $h \in \mathcal{H}^+(U)$,

$$\sup_{x\in K} h(x) \leqslant c_1 \inf_{x\in K} h(x) + c_2 \qquad \text{(HI)}$$

We remark that , if for every $\lambda > 0$ and $h \in \mathcal{H}^+(U)$ we have $\lambda h \in \mathcal{H}^+(U)$, then we can choose $c_2 = 0$ and we obtain the classical Harnack inequality .

The Harnack inequality , for quasilinear elliptic equation , is proved in the funda - mental tools of Serrin [1 9] , see also [20 , 1 3] . For the linear case see [9 , 3 , 1 , 8] .

In the rest of this section , we assume that $\mathcal{B}$ satisfy the following supplementary condition .

d

(∗) There exists $b \in L^{p-\varepsilon}_{\rm loc}(\mathbb{R}^d), 0 < \varepsilon < 1$, such that $| \mathcal{B}(x,\zeta) | \leqslant b(x) | \zeta |^\alpha$ for

every$x \in \mathbb{R}^d$and$\zeta \in \mathbb{R}$.

Small powers $(0 < \alpha < p-1)$. We have the validity of Harnack principle given by the following proposition .

Proposition 7 . 1 . *Let $\mathcal{H}$ be the sheaf of th e continuous s o lutions of the equation (1 . 1) . Assume that the condition $(*)$ is satisfies with $0 < \alpha < p-1$. Then the Harnack principle is satisfied by $\mathcal{H}$.*

The proof of this proposition can be found in [1 8 , p . 1 78] or [1 9]

Definition 7 . 1 . The sheaf $\mathcal{H}$ is called elliptic if for every regular domain V in $\mathbb{R}^d$, $x \in V$ and $f \in \mathcal{C}^+(\partial V), H_V f(x) = 0$ if and only if $f = 0$.

In the following example , we have the Harnack inequality but not the ellipticity . This is in contrast to the linear theory or quasilinear setting of nonlinear potential theory given by the $\mathcal{A}-$ harmonic functions in [1 0] .

Example 7 . 1 . We assume that $\mathcal{B}(x,\zeta) = \operatorname{sgn}(\zeta) \mid \zeta \mid^{\alpha}$ with $0 < \alpha < p-1$ and
$\mathcal{A}(x,\xi) = \mid \xi \mid^{p-2} \xi$. Let $u = cr^{\beta}$ with $\beta = p(p-1-\alpha)^{-1}$ and

$$c = p^{pp_{-}^{-1}{}_{1-\alpha}}(p-1-\alpha)pp_{-1-}\alpha[d(p-1-\alpha)+\alpha p]p-1^{1}_{-}\alpha.$$

With an easy verification , we will find that for every $x_0 \in \mathbb{R}^d$ and ball B (x_0,ρ), there exists a solution $u($ in the form $c \parallel x - x_0 \parallel \beta_)$ on B (x_0,ρ) such that $\Delta_p u = u^{\alpha}$ with $u(x_0) = 0$ and $u(x) > 0$ for every $x \in$ B $(x_0,\rho) \setminus \{x_0\}$. We therefore obtain that the sheaf $\mathcal{H}$ is not elliptic and curiously we have the existence of a basis of regular set $\mathcal{V}$ such that for every $V \in \mathcal{V}$, there exist $x_0 \in V$ and $f \in \mathcal{C}(\partial V)$ with $f > 0$ on

$$\partial V \text{and} H_V f(x_0) = 0.$$

We will prove that the sheaf given in the previous example is non - degenerate in the following sense :

Definition 7 . 2 . A sheaf $\mathcal{H}$ is called non - degenerate on an open U if for every $x \in U$, there exists a neighborhood V of x and $h \in \mathcal{H}(V)$ with $h(x) \neq 0$.

Proposition 7 . 2 . *Assume that th e condition $(*)$ is satisfies with $0 < \alpha < p-1$ and $\mathcal{A}(x,\lambda\xi) = \lambda \mid \lambda \mid^{p-2} \mathcal{A}(x,\xi)$ for al l $x, \xi \in \mathbb{R}^d$ and for al l $\lambda \in \mathbb{R}$. Then the sheaf $\mathcal{H}$ is non degenerate and more we have : for every regular s e t V and $x \in V$,*

$$\sup_{h \in \mathcal{H}(V)} h(x) = +\infty.$$

Proof . It is sufficient to prove that for every $x_0 \in \mathbb{R}^d, \rho > 0, n \in \mathbb{N}$ and $u_n = H_{B(x_0,\rho)} n$ we have u_n converges to infinity at any point of B (x_0,ρ). The comparison principle yields that $0 \leqslant u_n \leqslant n$ on B (x_0,ρ). Put $u_n = nv_n$, we then obtain :

$$\int \mathcal{A}(x,\nabla v_n)\nabla\phi dx + n^{1-p}\int \mathcal{B}(x,nv_n)\phi dx = 0$$

for every $\phi \in \mathcal{C}_c^{\infty}($ B $(x_0,\rho))$ and for every $n \in \mathbb{N}^*$. The assumptions on $\mathcal{B}$ yields

$$\lim_{n\to\infty}\int \mathcal{A}(x,\nabla v_n)\nabla\phi dx = 0;$$

since $0 \leqslant v_n \leqslant 1$, we have

$$|n^{1-p}\mathcal{B}(x,nv_n)| \leqslant n^{\alpha-p+1}b(x) \leqslant b(x)$$

and by [1 8 , Theorem 4.19]$, v_n$ are equicontinuous on the closure $B_{x_0,\rho}$ of the ball B (x_0,ρ), then by the Ascoli ' s theorem $, (v_n)_n$ admits a subsequence which is uniformly convergent on $B_{x_0,\rho}$ to a continuous function v on $B_{x_0,\rho}$. Further we can easily verify that $v \in \mathcal{W}^{1,p}_{\mathrm{loc}}(B(x_0,\rho))$ and

$$\int \mathcal{A}(x,\nabla v)\nabla\phi dx = 0$$

for every $\phi \in \mathcal{W}^{1,p}_0(B(x_0,\rho))$. Since $v = 1$ on $\partial B(x_0,\rho), v = 1$ on $B_{x_0,\rho}$. The relation $u_n = nv_n$ yields the desired result. □

Big Powers $(\alpha \geqslant p-1)$. We shall investigate (1.1) in the case $\alpha \geqslant p-1$. Let $\mathcal{H}$ be the sheaf of the continuous solutions of (1.1). In [18] or [19], we find the following form of the Harnack inequality.

Theorem 7.1. *Assume that the condition $(*)$ is satisfies with $\alpha \geqslant p-1$. Then For every non empty open set U in $\mathbb{R}^d$, for every constant $M > 0$ and every compact K in U, there exists a constant $C = C(K,M) > 0$ such that for every $u \in \mathcal{H}^+(U)$*

$$with u \leqslant M,$$
$$\sup_K u \leqslant C \inf_K u.$$

Corollary 7.1. *If the condition $(*)$ is satisfies with $\alpha \geqslant p-1$, then $\mathcal{H}$ is non-degenerate and elliptic. Moreover, for every domain U in $\mathbb{R}^d$ and $u \in \mathcal{H}^+(U)$, we have either $u > 0$ on U or $u = 0$ on U.*

Remark 7.1. If $\alpha = p-1$, the constant in *Theorem 7.1* does not depend on M and we have the classical form of the Harnack inequality.

We recall that a sheaf $\mathcal{H}$ satisfies the *Brelot convergence property* if for every domain U in $\mathbb{R}^d$ and for every monotone sequence $(h_n)_n \subset \mathcal{H}(U)$ we have $\lim_n h_n \in \mathcal{H}(U)$ if it is not identically $+\infty$ on U.

Using the same proof as in [4], we have the following proposition.

Proposition 7.3. *If the Harnack inequality is satisfied by $\mathcal{H}$, then the convergence property of Brelot is fulfilled by $\mathcal{H}$.*

Remark 7.2. In contrast to the linear case *(see [16])* the converse of *Proposition 7.3* is not true *(see [5])* and hence the validity of the convergence property of Brelot does not imply the validity of the Harnack inequality.

An Application. Let $\mathcal{H}_\alpha$ be the sheaf of all continuous solution of the equation

$$-\mathrm{div}\mathcal{A}(x,\nabla u) + b(x)\mathrm{sgn}(u) \mid u \mid^\alpha = 0$$
$$\text{where} b \in L^{d-\varepsilon \text{loc}}_{d}(\mathbb{R}^d), b \geqslant 0 \text{and} 0 < \varepsilon < 1.$$

Theorem 7.2. *a) For each $0 < \alpha < p-1$, $(\mathbb{R}^d, \mathcal{H}_\alpha)$ is a Bauer harmonic space satisfying the Brelot convergence property, but it is not elliptic in the sense of Definition 7.1.*
b) For each $\alpha \geqslant p-1, (\mathbb{R}^d, \mathcal{H}_\alpha)$ is a Bauer harmonic space elliptic in the sense of Definition 7.1 and the convergence property of Brelot is fulfilled by $\mathcal{H}_{p-1}$.

8. Keller - Osserman Property

Let $\mathcal{H}$ be the sheaf of continuous solutions related to the equation (1.1). **Definition 8.1.** Let U be a relatively compact open subset of $\mathbb{R}^d$. A function $u \in \mathcal{H}^+(U)$ is called regular Evans function for $\mathcal{H}$ and U if $\lim^{u(x)}_{\ni_U x \to z} = +\infty$ for every regular point z in the boundary of U.

For an investigation of regular Evans functions see [5].

Definition 8.2. We shall say that $\mathcal{H}$ satisfies the Keller-Osserman property, denoted (KO), if every ball admits a regular Evans function for $\mathcal{H}$.

As in [5, Proposition 1.3], we have the following proposition.

Proposition 8.1. *$\mathcal{H}$ satisfies the (KO) condition if and only if $\mathcal{H}^+$ is locally uniformly bounded (i.e. for every non empty open set U in $\mathbb{R}^d$ and for every compact $K \subset U$, there exists a constant $C > 0$ such that $\sup_K u \leqslant C$ for every $u \in \mathcal{H}^+(U)$).*

Corollary 8.1. *If $\mathcal{H}$ fulfills the (KO) property, then $\mathcal{H}$ satisfies the Brelot convergence property.*

Theorem 8.1. *Assume that $\mathcal{A}$ and $\mathcal{B}$ satisfies the following supplementary conditions*

i) *For every $x_0 \in \mathbb{R}^d$, the function F from $\mathbb{R}^d$ to $\mathbb{R}^d$ defined by $F(x) = \mathcal{A}(x, x - x_0)$ is differentiable and* div F *is locally (essentially) bounded.*

ii) $\mathcal{A}(x, \lambda\xi) = \lambda |\lambda|^{p-2} \mathcal{A}(x, \xi)$ *for every $\lambda \in \mathbb{R}$ and every $x, \xi \in \mathbb{R}^d$.*

iii) $|\mathcal{B}(x, \zeta)| \geqslant b(x) |\zeta|^{\alpha}$, $\alpha > p-1$ *where* $b \in L^{d-\varepsilon}_{d}{}^{loc}(\mathbb{R}^d)$, $0 < \varepsilon < 1$, *with* $\mathrm{ess}_U \inf b(x) > 0$ *for every relatively compact U in $\mathbb{R}^d$.*

Then the (KO) property is valid by $\mathcal{H}$.

Proof. Let U be the ball with center $x_0 \in \mathbb{R}^d$ and radius R. Put $f(x) = R^2 - \| x - x_0 \|^2$ and $g = cf^{-\beta}$, we obtain the desired property if we find a constant $c > 0$ such that g is a supersolution of the equation (1.1). We have $\nabla f(x) = -2(x - x_0)$ and $\nabla g(x) = 2c\beta(f(x))^{-(\beta+1)}(x - x_0)$ and then

$$\mathcal{A}(x, \nabla g(x)) = (2c\beta)^{p-1}(f(x))^{-(\beta+1)(p-1)}\mathcal{A}(x, x - x_0).$$

Let $\phi \in \mathcal{C}_c^{\infty}(U), \phi \geqslant 0$ and we set $I_\phi = \int \mathcal{A}(x, \nabla g)\nabla\phi dx + \int \mathcal{B}(x, g)\phi dx$, then

$$\begin{aligned}
I_\phi &= -\int \mathrm{div}\mathcal{A}(x, \nabla g)\phi dx + \int \mathcal{B}(x, g)\phi dx \\
&= -\int [2(\beta+1)(p-1)(2c\beta)^{p-1} f^{-(\beta+1)(p-1)-1}\mathcal{A}(x, x - x_0).(x - x_0) \\
&\quad + (2c\beta)^{p-1} f^{-(\beta+1)(p-1)}\mathrm{div}\mathcal{A}(x, x - x_0) - \mathcal{B}(x, g)]\phi dx \\
&\geqslant -\int [2(\beta+1)(p-1)(2c\beta)^{p-1} f^{-(\beta+1)(p-1)-1}\mathcal{A}(x, x - x_0).(x - x_0) \\
&\quad + (2c\beta)^{p-1} f^{-(\beta+1)(p-1)}\mathrm{div}\mathcal{A}(x, x - x_0) - c^{\alpha} b f^{-\alpha\beta}]\phi dx \\
&= -\int [2c^{p-1-\alpha}(2\beta)^{p-1}(\beta+1)(p-1)\mathcal{A}(x, x - x_0).(x - x_0) \\
&\quad + c^{p-1-\alpha}(2\beta)^{p-1} f \mathrm{div}\mathcal{A}(x, x - x_0) - b f^{\beta(p-1-\alpha)+p}] c^{\alpha} f^{-(\beta+1)(p-1)-1}\phi dx.
\end{aligned}$$

Putting $\beta = p(\alpha - p + 1)^{-1}$ we obtain

$$\begin{aligned}
I_\phi \geqslant &-\int [2(\tfrac{2p}{\alpha-p+1})^{p-1}(\tfrac{\alpha+1}{\alpha-p+1})(p-1)\mathcal{A}(x, x - x_0).(x - x_0) \\
&+ (\tfrac{2p}{\alpha-p+1})^{p-1} f \mathrm{div}\mathcal{A}(x, x - x_0) - c^{\alpha-p+1} b] c^{p-1} f^{\frac{p\alpha}{p-1-\alpha}}\phi dx.
\end{aligned}$$

 It follows from A 2 that $\mathcal{A}(x, x-x_0).(x-x_0)$ is lo cally bounded . Hence if we take

$c$sothat$^{p-1}_{\alpha}{}_{-p+1}$

$$c \geqslant [_x\sup_{\in U}\{2(\alpha^{+1)(p-1)}_{\alpha}{}_{-p+1} \mid \mathcal{A}(x, x-x_0 b^{).(x}_{(x)} - x_0) \mid +R^2 \mid \operatorname{div}\mathcal{A}(^{x,x}_{b\ (x)} - x_0) \mid\}]1_{-\alpha p+}1$$

$$\times(_\alpha 2^p_{-p}+1)p^{-1}_{\alpha-p+}1,$$

then $I_\phi \geqslant 0$ holds for every $\phi \in \mathcal{C}^\infty_c(U)$ with $\phi \geqslant 0$. Thus the function $g(x) = c(R^2 - \parallel x - x_0 \parallel^2)^{p(p-1-\alpha)}$ is a supersolution satisfying $\lim_{x\to z} g(x) = +\infty$ for every $z \in \partial U$. By the comparison principle we have H $U^n \leqslant g$ for every $n \in \mathbb{N}$ and therefore , the increasing sequence $(H_U n)_n$ of harmonic functions is lo cally uniformly bounded on U. The Bauer convergence property implies that $u = \sup H_U n \in \mathcal{H}(U)$,

n

therefore we have $\lim_{x\to z} \inf u(x) \geqslant n$ for every z in ∂U, thus $\lim_{x\to z} u(x) = +\infty$ for every z in ∂U and u is a regular Evans function . Since U is an arbitrary ball , we get the desired property . □

Corollary 8 . 2 . *Under the assumptions in Theorem 8 . 1 , for every ball* B = B (x_0, R) *with center* x_0 *and radius* R *and for every* $u \in \mathcal{H}(U)$,

$$\mid u(x_0) \mid \leqslant cR2^p_{-p}1-\alpha$$

where

$$c = [\sup_{x\in B}\{2(\alpha^{+1)(p-1)}_{\alpha}{}_{-p+1} \mid \mathcal{A}(x, x-x_0 b^{).(x}_{(x)} - x_0) \mid +R^2 \mid \operatorname{div}\mathcal{A}(^{x,x}_{b\ (x)} - x_0) \mid\}]1_{p_{-\alpha}}+1$$

$$\times(_\alpha 2^p_{-p}+1)p^{-1}_{\alpha-p+}1.$$

Proof . From the proof of the previous theorem , if $B_n =$ B $(x_0, R(1-n^{-1})), n \geqslant 2$, we have

$$u(x_0) \leqslant c_n \left(\begin{array}{c} R(n-1) \\ n \end{array}\right) 2^p_{-p}1-\alpha$$

for every $n \geqslant 2$ and

$$\begin{aligned} c_n \quad &= \quad \Big[\begin{array}{c} \sup \\ x \in B_n \end{array} \Big\{ 2(\alpha^{+1)(p-1)}_{\alpha}{}_{-p+1} \mid \mathcal{A}(x, -x_0 b^{).(x}_{(x)} - x_0) \mid \\ &+ \left(\begin{array}{c} R(n-1) \\ n \end{array}\right)^2 \mid \operatorname{div}\mathcal{A}(^{x,x}_{b\ (x)} - x_0) \mid\}]1_{-\alpha p+}1 \quad \left(\begin{array}{c} 2p \\ \alpha - p + 1 \end{array}\right) p^{-1}_{\alpha-p+1} \\ &\leqslant \quad \Big[\begin{array}{c} \sup \\ x \in B \end{array} \Big\{ 2(\alpha^{+1)(p-1)}_{\alpha}{}_{-p+1} \mid \mathcal{A}(x, x-x_0 b^{).(x}_{(x)} - x_0) \mid \\ &+R^2 \mid \operatorname{div}\mathcal{A}(^{x,x}_{b\ (x)} - x_0) \mid\}]1_{-\alpha p+}1 \quad \left(\begin{array}{c} 2p \\ \alpha - p + 1 \end{array}\right) p^{-1}_{\alpha-p+1}. \end{aligned}$$

Then we obtain the inequality

$$u(x_0) \leqslant cR2^p_{-p}1-\alpha.$$

Since $-u$ is a solution of similarly equation , we get

$$-u(x_0) \leqslant cR2^p_{-p}1-\alpha$$

with the same constant c as before . Then we have the desired inequality . ☐ We now have a Liouville like theorem .

Theorem 8 . 2 . *Assume that the conditions in* Theorem 8 . 1 *are satisfied and that*

$$\lim_{R\to\infty}\inf(R^{-2p}M(R))=0$$

where

$$M(R)=\quad \| \sup_{-x x_0\|} \leqslant R\{2(\alpha_\alpha^{+1)(p-1)}{}_{-p+1} \mid \mathcal{A}(x,x-x_0 b^{).(x}_{(x)}-x_0)\mid +R^2 \mid \mathrm{div}\mathcal{A}(^{x,x}_{b\ (x)}-x_0)\mid\}.$$

Then $u\equiv 0$ *is the unique s o lutio n of th e equation (1 . 1) on* $\mathbb{R}^d$. *Proof .* Let u be a solution of the equation (1 . 1) on $\mathbb{R}^d$. By the previous corollary ,
we have for every $x_0\in\mathbb{R}^d$ and every $R>0$

$$\mid u(x_0)\mid \quad\leqslant\quad [_{\|}\sup_{-x x_0\|}\leqslant R\begin{cases} 2(\alpha+1)(p-1)\mid \mathcal{A}(x,x-x_0).(x-x_0)\mid \\ \alpha-p+1 \quad b(x)\end{cases}$$

$$+R^2\mid \mathrm{div}\mathcal{A}(^{x,x}_{b\ (x)}-x_0)\mid\}R^{-2p}]1_{-\alpha p+}1 \quad \begin{pmatrix} 2p \\ \alpha-p+1\end{pmatrix} p-minus^1_{\alpha-p+1}.$$

Hence$u(x_0)=0$and$u\equiv 0$. ☐

9 . Applications

We shall use the previous results for the investigation of the $p-minus$ Laplace $\Delta_p, p\geqslant 2$
which is the Laplace operator if $p=2$. Δ_p is associated with $\mathcal{A}(x,\xi)=\mid\xi\mid^{p-2}\xi$,
an easy calculation gives div $\mathcal{A}(x,x-x_0)=(d+p-2)\parallel x-x_0\parallel p-2$. Let , for every $\alpha>0,\mathcal{H}_\alpha$ denote the sheaf of all continuous solution of the equation

$$-\Delta_p u+b(x)\mathrm{sgn}(u)\mid u\mid^\alpha=0 \quad (9.1)$$

$$d$$

where$b\in L^{d-\varepsilon}_{\mathrm{loc}}(\mathbb{R}^d), b\geqslant 0and0<\varepsilon<1$.

Theorem 9 . 1 . *Assume that* $p\geqslant 2$. *For* $\alpha>0$, *le t* $\mathcal{H}_\alpha$ *denote the sheaf of al l continuous s o lutio n of th e equation*

$$-\Delta_p u+b(x)\mathrm{sgn}(u)\mid u\mid^\alpha=0.$$

$$d$$

where $b\in L^{d-\varepsilon}_{loc}(\mathbb{R}^d), b\geqslant 0$ *and* $0<\varepsilon<1$. *Then*

(1) *For every* $\alpha>0$, $(\mathbb{R}^d,\mathcal{H}_\alpha)$ *is a nonlinear Bauer harmonic space with the Brelot convergence Property .*

(2) $\mathcal{H}_\alpha$ *is e l lip ti c for e very* $\alpha\geqslant p-1$.

(3) *If* $\alpha>p-1$ *and* $\inf_U b>0$ *for every relatively compact open* U *in* $\mathbb{R}^d$, *then the property (KO) is satisfied b y* $\mathcal{H}_\alpha$.

(4) *If* $\alpha>p-1$ *and* $,\inf_{\mathbb{R}} db>0$, *th en* $\mathcal{H}_\alpha(\mathbb{R}^d)=\{0\}$.

Theorem 9 . 2 . *Let* $U\subset\mathbb{R}^d$ *be an bounded open s e t whose boundary ,* ∂U, *can be represented locally as a graph of function with H ö lder continuous derivatives . Assume that* $\alpha>p-1$. *Then* U *admits a regular Evans function for* $\mathcal{H}$.

 Proof
. We first prove the existence of a continuous supersolution v on U such that

$$\lim_{x\to z} v(x) = +\infty, \text{foreevery} z \in \partial U.$$

Let f in $\mathcal{C}_c^\infty(U)$ be a positive function ($f \neq 0$) and $w \in \mathcal{W}_0^{1,p}(U)$ be the solution of the problem

$$\int_U \mid \nabla w \mid^{p-2} \nabla w \cdot \nabla \phi dx = \int_U f\phi dx, \quad \phi \in \mathcal{W}_0^{1,p}(U)$$
$$w = 0 \quad \text{on} \partial U$$

By the regularity theory , w has a H ö lder continuous gradient , w is continuous supersolution $w > 0$ in $U, \lim_{x\to z} w(x) = 0$ for every $z \in \partial U$ and $\| w \| \infty^+ \| \nabla w \| \infty \to 0$ as $\| f \| \infty \to 0$. Then we set $v = w^{-\beta}$ and look for $\beta > 0$ and f such that

$$\int_U \mid \nabla v \mid^{p-2} \nabla v \cdot \nabla \phi dx + \int_U b(x) v^\alpha \phi dx \geqslant 0 \quad \phi \geqslant 0, \phi \in \mathcal{W}_0^{1,p}(U).$$

For every $\phi \geqslant 0, \in \mathcal{W}_0^{1,p}(U)$, we have

$$\begin{aligned}
\int_U \mid \nabla v \mid^{p-2} \nabla v \cdot \nabla \phi dx &= -\beta^{p-1} \int_U w^{-(\beta+1)(p-1)} \mid \nabla w \mid^{p-2} \nabla w \cdot \nabla \phi dx \\
&= -\beta^{p-1} \int_U \mid \nabla w \mid^{p-2} \nabla w \cdot \nabla (w^{-(\beta+1)(p-1)} \phi) dx \\
&\quad -\beta^{p-1}(\beta+1)(p-1) \int_U w^{-(\beta+1)(p-1)-1} \phi \mid \nabla w \mid^p dx \\
&= -\beta^{p-1} \int_U w^{-(\beta+1)(p-1)-1} [wf + (\beta+1)(p-1) \mid \nabla w \mid^p] \phi dx;
\end{aligned}$$

thus

$$\int_U \mid \nabla v \mid^{p-2} \nabla v \cdot \nabla \phi dx$$
$$+\beta^{p-1} \int_U bv(\beta+1)(_\beta p-1)+1 \quad [b^{-1} wf + (\beta+1)(p-1) b^{-1} \mid \nabla w \mid^p] \phi dx = 0.$$

Put $\beta = p_{\alpha-p+1}$ and choose f such that $wf + (\beta+1)(p-1) \mid \nabla w \mid^p \leqslant b\beta^{1-p}$. Then

$$\int_U \mid \nabla v \mid^{p-2} \nabla v \cdot \nabla \phi dx + \int_U bv^\alpha \phi dx \geqslant 0, \quad \text{forevery} \phi \geqslant 0, \phi \in \mathcal{W}_0^{1,p}(U);$$

therefore , v is a continuous supersolution of (9 . 1) such that $\lim_{x\to z} v(x) = +\infty$, for

$$\text{every} z \in \partial U.$$

Let u_n denote the continuous solution of the problem

$$\int_U \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla \phi dx + \int_U bu^\alpha \phi dx = 0, \quad \phi \in \mathcal{W}_0^{1,p}(U)$$
$$u = n \in \mathbb{N} \quad \text{on} \partial U$$

By the comparison principle we have $0 \leqslant u_n \leqslant v$ for all n and by the convergence property , the function $u = \sup_n u_n$ is a regular Evans function for $\mathcal{H}$ and U. □

Theorem 9 . 3 . *Let* $\alpha > p - 1$ *and le t* U *be a s tar domain and* b *continuous and s trictly positive function on* $\mathbb{R}^d$. *Assume that th e conditions in Theorem 9 . 1 are satisfied . If th ere exists a regular Evans function* u *associated with* U *and* $\mathcal{H}_\alpha$, *then* u *is unique .*

The proof is the same as in [4] and [6] when $b \equiv 1$.

References

[1] M . Aisenman and B . Simon , *Brownian motion and Harnack inequality for Schr ö dinger op - erators* , Comm . Pure Appl . Math . (1 982) , no . 35 , 209 – 273 .

[2] N . BelHadj Rhouma , A . Boukricha , and M . Mosbah , *Perturbations et espaces harmoniques nonlin é aires* , Ann . Academiae Scientiarum Fennicae (1 998) , no . 23 , 33 – 58 .

[3] A . Boukricha , W . Hansen , and H . Hueber , *Continuous so lutions of the generalized Schr ö dinger equation and perturbation of harmonic spaces* , Exposition . Math . **5** (1 987) , 97 – 1 35 .

[4] A . Boukricha , *Harnack inequality for nonlinear harmonic spaces* , Math . Ann . **317** (2000) 3 , 567 – 583 .

[5] A . Boukricha , *Keller - Osserman condition and regular Evans functions for s emilinear PDE* , Preprint .

[6] E . B . Dynkin , *A pro bab ilistic appraoch to one cl ass of nonlinear differential equations* , Prob . The . Rel . Fields (1 99 1) , 89 – 1 1 5 .

[7] F . A . van Gool , *Topics in nonlinear potential theory* , Ph . D . thesis , September 1 992 .

[8] D Gilbarg and N . S . Trudinger , *El l ip tic partial differential equations of second order* , second ed . , Die Grundlehren der Mathematischen Wissenschaften , no . 224 , Springer - Verlag , Berlin , 1 983 .

[9] W . Hansen , *Harnack inequalities for Schroedinger operators* , Ann . Sc . Norm . Super . Pisa , Cl . Sci . , IV . Ser . 28 , No . 3 , 41 3 – 470 (1 999) .

[10] J . Heinonen , T . Kilpl ä inen , and O . Martio , *Nonlinear potential theory of degenerate el liptic equations* , Clarendon Press , Oxford New York Tokyo , 1 993 .

[1 1] M . A . Krasnosel ' ski ĭ, *Topological methods in theory of nonlinear integral equations* , Pergamon Press , 1 964 .

[1 2] D . Kinderlehrer and G . Stampacchia , *An introduction to variational inequalities and their applications* , Academic Press , New York , 1 980 .

[1 3] P . Lehtola , *An axiomatic approch to nonlinear potential theory* , Ann . Academiae Scientiarum Fennicae (1 986) , no . 62 , 1 – 42 .

[14] J . L . Lions , *Quellques m é thodes de r é so lution des probl è mes aux l imites nonlin é aires* , Dunod Gautheire - Villars , 1 969 .

[1 5] O . A .Ladyzhens^{k-a} ya and N . N . Ural ' tseva , *Linear and quasilinear ellip tic equations* , Math - ematics in Science and Engineering , no . 46 , Academic Press , New York , 1 968 .

[1 6] P . A . Loeb and B . Walsh , *The equivalence of Harnack ' s principle and Harnack ' s inequality in the axiomatic system of Brelot* , Ann . Inst . Fourier **1 5** (1 965) , no . 2 , 597 – 600 .

[17] Fumi - Yuki Maeda , *Semilinear perturbation of harmonic spaces* , Hokkaido Math . J . **1 0** (1 98 1) , 464 – 493 .

[18] J . Mal ý and W . P . Ziemmer , *Fine regularity of s olutions of partial differential equations* , Mathematical Surveys and monographs , no . 5 1 , American Mathematical Society , 1 997 .

[1 9] J . Serrin , *Local behavior of s olutions of quasilinear equations* , Acta Mathematica (1 964) , no . 1 1 , 247 – 302 .

[20] N . S . Trudinger , *On Harnack type inequality and their application to quasilinear el liptic equations* , Comm . Pure Appl . Math . (1 967) , no . 20 , 721 – 747 .

Azeddine Baalal

Dé partement de Math é matiques et d ' Informatique , Facult é des Sciences A ï n Chock , Km 8 Route El Jadida B . P . 536 6 M â arif , Casablanca - Maroc

E - mail address : baalal @ facsc - achok . ac . ma

Abderahman Boukricha

D é partement de Math é matiques , Facult é des Sciences de Tunis , , Campus Universitaire 1 0 60 Tunis - Tunisie .

E - mail address : aboukr i cha @ f st . rnu . tn