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Bursters and quasi-periodic solutions of a self-excited quasi-periodic Mathieu oscillator

Faouzi Lakrad [a,*], Abderrahim Azouani [b], Nazha Abouhazim [c], Mohamed Belhaq [c]

[a] Institut Polytechnique, Rue 10, Numéro 2, Lotissement Allaimoun 1, Casablanca, Morocco
[b] Institute of Mathematics I, Free University Berlin, Arnimallee 2-6, D-14195 Berlin, Germany
[c] Laboratory of Mechanics, Faculty of Sciences Ain Chock, PB 5366 Maârif, Casablanca, Morocco

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Abstract

In this paper the conditions of occurrence of quasi-periodic (QP) solutions and bursting dynamics in a self-excited quasi-periodic Mathieu Oscillator are discussed. The quasi-periodic excitation consists of two periodic excitations; one with a very slow frequency and the other with a frequency resonant with the proper frequency of the oscillator. The fast dynamics are initially averaged. The complimentary quasi-static solutions of the modulation equations of amplitude and phase are determined and their stability is analyzed. Numerical simulations and power spectra are shown to complete the theoretical analysis.

1. Introduction

Quasi-periodically forced systems are systems that are influenced by two periodic signals with incommensurate frequencies. These systems have received much interest in the last decade because of the complicated dynamics which they involve and the invariant sets that they can support. 'Quasi-periodic driven' implies that the most elementary invariant sets are tori. In addition, strange invariant sets can occur. The strange non-chaotic attractors [1] and the chaotic attractors are pointed out.

Heagy and Ditto [2] investigated the transition from two-frequency quasi-periodicity to chaotic behavior in a model for a quasi-periodically driven magnetoelastic ribbon. The model system was a two-frequency parametrically driven Duffing oscillator. They found that the transition to chaos takes four stages as the driven parameter is changed: torus doubling, strange non-chaotic attractor, geometrically similar chaotic attractor, and a crisis leading to chaos. Belogortsev [3] analytically studied the tangent and doubling bifurcations of tori in the weakly non-linear Duffing oscillator driven by a two-periodic external force. Stupnicka and Rudowski [4] numerically and theoretically studied the behavior of the van der Pol-Duffing periodically forced oscillator at the passage through principal resonance. Almost periodic

* Corresponding author.
E-mail address: lakrad@recherche-maroc.org (F. Lakrad).

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oscillations, frequency locking, transition to chaotic motion, and the jump from the non-resonant to the resonant state were observed and interpreted. Yagasaki [5] studied, using the Melnikov theory, the intersections of the stable and unstable manifolds of the normally hyperbolic invariant torus of a two-frequency perturbation of Duffing's equation. Maccari [6] used a perturbation method to study the quenching effect of multiple resonant parametric periodic excitations on a generalized non-linear oscillator.

A special interest has been given to the quasi-periodically forced Mathieu oscillators. Hence, Broer and Simó [7] explored geometrically resonance tongues containing instability pockets in a linear Hill's equation with quasi-periodic forcing. Rand and co-workers in [8] and [9] determined an approximation of the regions of stability using four different methods: direct numerical integration, Lyapunov exponents, regular perturbations, and harmonic balance. They also investigated the interaction of subharmonic resonance bands using Chirikov's overlap criterion [10] and the transition from local chaos to global chaos. Belhaq and co-workers [11,12] analytically approximated QP solutions and studied the stability of a damped cubic non-linear QP Mathieu equation, using a double perturbation method.

This paper considers a self-excited quasi-periodic Mathieu oscillator or a van der Pol-Mathieu QP oscillator of the form

$$\ddot{x} + \omega^2 x = -[(\rho\cos(\nu t) + h\cos(\Omega t))x + (-\alpha + \beta x^2)\dot{x}] \tag{1}$$

Eq. (1) represents a four-dimensional dynamical system in the phase space $\mathbb{R}^2 \times \mathbb{T}^2$ as can be seen by writing the system as

$$\dot{x} = y \tag{2}$$

$$\dot{y} = -\omega^2 x - [(\rho\cos(\theta_1) + h\cos(\theta_2))x + (-\alpha + \beta x^2)y] \tag{3}$$

$$\dot{\theta}_1 = \nu \tag{4}$$

$$\dot{\theta}_2 = \Omega \tag{5}$$

The unforced version of Eq. (1) i.e.; $\rho = h = 0$ is the classical van der Pol equation and has a unique limit cycle. One should expect within the regular behavior of systems of the form (1) the occurrence of 3-period-QP solutions. Two frequencies should be related to the quasi-periodic excitation and one to the self-excitation induced by the term $(\alpha - \beta x^2)\dot{x}$.

A version of Eq. (1) was recently analyzed by Abouhazim et al. [13] in the case of $\Omega = \mathcal{O}(\varepsilon)$ and ν resonant with the proper frequency ω. They focused on the construction and the determination of existence and stability of 3-period-QP solutions. They used the so-called *double perturbation* method [14]. This method uses two perturbation parameters to naturalize the application of two reductions through the multiple scales method (MSM) [15]. The periodic solutions of the second reduced system, which is autonomous, correspond to 3-period-QP solutions of the original oscillator.

This paper considers that the QP parametric excitation consists of a resonant frequency ν with the proper frequency ω and a *very slow* frequency $\Omega = \mathcal{O}(\varepsilon^p)$ where p is an integer $\geqslant 2$. This very slow excitation induces quasi-static solutions on the slow manifold resulting from an averaging over the fast scale of time. Consequently, a change in the nature of the quasi-static solutions during a period of the very slow frequency, leads to the appearance of periodic bursters. Thus, in this work the conditions of existence and the construction of QP solutions and periodic bursters solutions are focused on. For detailed classification of bursters see Golubitsky et al. [16].

The following topics are covered in this paper; in Section 2 an averaging over the fast dynamics is performed; in Section 3 the regular dynamics of Eq. (1) in the absence of the very slow parametric excitation i.e., $h = 0$ is analyzed; in Section 4 the effects of the very slow parametric excitation are discussed. Numerical simulations and power spectra are shown to complete the theoretical analysis.

2. Perturbation analysis

To analytically examine the response of the self-excited QP Eq. (1), a parameter of perturbation ε is introduced, and the parameters of Eq. (1) are scaled as follows: $\alpha = \varepsilon\tilde{\alpha}$, $h = \varepsilon\tilde{h}$, $\beta = \varepsilon\tilde{\beta}$, $\rho = \varepsilon\tilde{\rho}$, $\Omega = \varepsilon^p\tilde{\Omega}$ and $\tau = \varepsilon^p t$. Eq. (1) is then rewritten as

$$\ddot{x} + \omega^2 x = -\varepsilon[(\tilde{\rho}\cos(\nu t) + \tilde{h}\cos(\tilde{\Omega}\tau))x + \tilde{\alpha}\dot{x} - \tilde{\beta}x^2\dot{x}] \tag{6}$$

The analysis is restricted to regular motions in the vicinity of the primary resonance one-half which is expressed as

$$\nu = 2\omega + \sigma \tag{7}$$

where $\sigma = \varepsilon\tilde{\sigma}$ is a detuning parameter. Using the multiple-scales method (MSM) [15], an approximate solution can be expressed in the form

$$x(t) = x_0(T_0, T_1, \tau) + \varepsilon x_1(T_0, T_1, \tau) + \mathcal{O}(\varepsilon^2) \tag{8}$$

where $T_0 = t$ is a fast time scale, $T_1 = \varepsilon t$ is a slow time scale describing the envelope of the response, and $\tau = \varepsilon^p t$ is a very slow time scale. In terms of the variable T_n, the time derivative becomes $\mathrm{d}/\mathrm{d}t = D_0 + \varepsilon D_1 + \mathcal{O}(\varepsilon^2)$, where $D_n = \partial/\partial T_n$.

Substituting Eq. (8) into Eq. (6), using Eq. (7) and equating coefficients of like powers of ε, we obtain the following hierarchy of problems :

$$\text{order } \mathcal{O}(1): \; D_0^2 x_0 + \omega^2 x_0 = 0 \tag{9}$$

$$\text{order } \mathcal{O}(\varepsilon): \; D_0^2 x_1 + \omega^2 x_1 = -2(D_0 D_1 x_0) - x_0(\tilde{\rho}\cos(\nu T_0) + \tilde{h}\cos(\widetilde{\Omega}\tau)) - (-\tilde{\alpha} + \tilde{\beta} x_0^2)(D_0 x_0) \tag{10}$$

The solution of Eq. (9) is

$$x_0(T_0, T_1, \tau) = A(T_1, \tau)\exp(\mathrm{i}\omega T_0) + cc \tag{11}$$

where cc denotes the complex conjugate of the preceding terms. The quantity $A(T_1, \tau)$ is to be determined by eliminating the secular terms at the next level of approximations. Substituting Eq. (11) into Eq. (10) and eliminating secular terms leads

$$\mathrm{i}2\omega(D_1 A) = -\frac{\tilde{\rho}}{2}\overline{A}\mathrm{e}^{\mathrm{i}\tilde{\sigma}T_1} - \tilde{h}A\cos(\widetilde{\Omega}\tau) + \mathrm{i}\omega(\tilde{\alpha}A - \tilde{\beta}A^2\overline{A}) \tag{12}$$

where the overbar indicates the complex conjugate. The particular solution of Eq. (10) is then written as

$$x_1(T_0, T_1, \tau) = -\frac{\tilde{\rho}}{2(\omega^2 - (\nu + \omega)^2)} A\mathrm{e}^{\mathrm{i}(\omega+\nu)T_0} + \mathrm{i}\frac{\tilde{\beta}}{8\omega}A^3\mathrm{e}^{\mathrm{i}3\omega T_0} + cc \tag{13}$$

Letting in Eq. (12) $A = \frac{1}{2}a\mathrm{e}^{\mathrm{i}\theta}$ where a and θ are real functions, separating real and imaginary parts, the slow flow modulation equations of amplitude a and phase γ are obtained by

$$\frac{\mathrm{d}a}{\mathrm{d}T_1} = -\frac{\tilde{\rho}}{4\omega}a\sin(2\gamma) + \frac{\tilde{\alpha}}{2}a - \frac{\tilde{\beta}}{8}a^3 \tag{14}$$

$$a\frac{\mathrm{d}\gamma}{\mathrm{d}T_1} = \frac{\tilde{\sigma}}{2}a - \frac{\tilde{\rho}}{4\omega}a\cos(2\gamma) - \frac{\tilde{h}}{2\omega}a\cos(\widetilde{\Omega}\tau) \tag{15}$$

where $\gamma = \frac{1}{2}\tilde{\sigma}T_1 - \theta$. One should point out that these equations are non-autonomous due to the presence of the term due to the very slow parametric excitation $\cos(\widetilde{\Omega}\tau)$. Furthermore, one should notice that the modulations of amplitude and phase are governed by a time scale T_1 which is still faster than the very slow time scale τ. Hence, one can conclude that the dynamics of Eq. (1) is governed by three time scales.

An approximation of the solution of Eq. (6), up to order $\mathcal{O}(\varepsilon^2)$ is given by combining the solution of order $\mathcal{O}(1)$ given in Eq. (11) and the solution of order $\mathcal{O}(\varepsilon)$ given in Eq. (13)

$$x(t) = a(\tau, T_1)\cos\left(\frac{\nu}{2}t - \gamma(\tau, T_1)\right) - \frac{\rho}{2(\omega^2 - (\nu + \omega)^2)}a(\tau, T_1)\cos\left(\frac{3\nu}{2}t - \gamma(\tau, T_1)\right) - \frac{\beta}{32\omega}a^3(\tau, T_1)\sin\left(\frac{3\nu}{2}t - 3\gamma(\tau, T_1)\right) + \mathcal{O}(\varepsilon^2) \tag{16}$$

In order to have a complete picture of the regular dynamics of the QP van der Pol–Mathieu Eq. (1) we study the modulation equations (14) and (15).

3. Stationary solutions

The modulation equations of amplitude and phase (14) and (15) have at most three fixed points and a limit cycle as stationary solutions.

- *Trivial solution*: $a = 0$ is all the time unstable. It is an unstable focus when it is the only fixed point of Eqs. (14) and (15), and an unstable node and/or a saddle when it coexists with non-trivial fixed points.

- *Non-trivial solutions*: $a_\pm$

$$a_\pm^2 = \frac{8}{\tilde{\beta}}\left[\frac{\tilde{\alpha}}{2} \pm \sqrt{H(\tau)}\right] \tag{17}$$

$$\gamma_\pm = \frac{1}{2}\arctan\left(\frac{-(\tilde{\alpha}/2) + (\tilde{\beta}/8)a_\pm^2}{-(\sigma/2) + (\tilde{h}/2\omega)\cos(\widetilde{\Omega}\tau)}\right) \tag{18}$$

$$H(\tau) = \frac{\tilde{\rho}^2}{16\omega^2} - \left(-\frac{\sigma}{2} + \frac{\tilde{h}}{2\omega}\cos(\widetilde{\Omega}\tau)\right)^2 \tag{19}$$

The solution a_+ is stable when it exists and a_- is unstable when it exists.
- *Limit cycle*: It is stable when it exists and is due to the van der Pol term in Eq. (1). It exists in the regions of parameters where the non-trivial solutions $a_\pm$ do not exist.

A usual approach in studying such systems is based on the so-called quasi-steady state assumption meaning that the fast variables are in a quasi-steady state i.e., the fixed points are no longer static points, but they depend on the very slow excitation $\cos(\widetilde{\Omega}\tau)$. This term has been considered as constant during the perturbation analysis. For more details about this approach see [17] and the reference therein. The dependence with respect to the very slow time scale can enable the solutions to cross the boundaries between different behaviors during one period $2\pi/\Omega$. This causes the periodic bursters to develop as will be shown.

3.1. Fixed points

3.1.1. Case h = 0

In order to illustrate the effect of the slow frequency parametric excitation, consider first the case where $h = 0$ i.e., the system (1) is only periodically forced by the resonant excitation $\rho\cos(\nu t)$. The non-trivial amplitudes are constants and are expressed in (17).

Fig. 1 shows that in the zone *I* only the unstable trivial focus exists, in the zone *II* the two non-trivial solutions also exist and in the zone *III* only a_+ coexists with the unstable trivial saddle solution. It is worth noting that the zone *II* where a_- exists shrinks to zero as $\alpha \to 0$.

It will be shown in the next section that the zone *I* corresponds to the zone of existence of a limit cycle.

In Fig. 2 the stationary amplitudes $a_\pm$ are shown versus the amplitude of the resonant parametric excitation ρ. The stable non-trivial solution is born through a saddle-node bifurcation and it increases while increasing the amplitude of

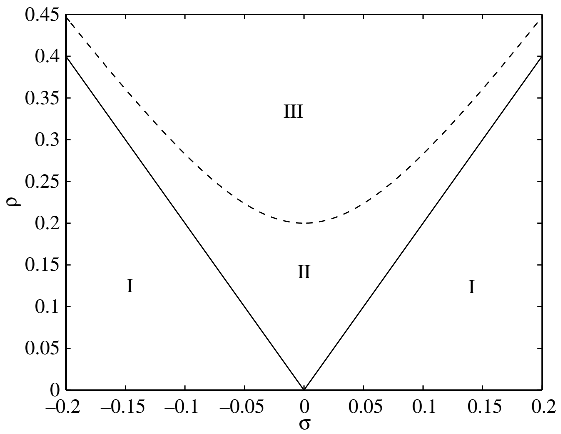

Fig. 1. Charts of behaviors of the modulation equations (14) and (15) in the absence of the very slow parametric excitation i.e., $h = 0$ and for $\alpha = 0.1$ and $\omega = 1$. In the zone *I* coexistence of an unstable trivial solution and a stable limit cycle. In the zone *II* coexistence of the unstable trivial fixed point and two non-trivial fixed points, one among them is stable. In zone *III* only the stable non-trivial fixed point and the unstable trivial solution coexist.

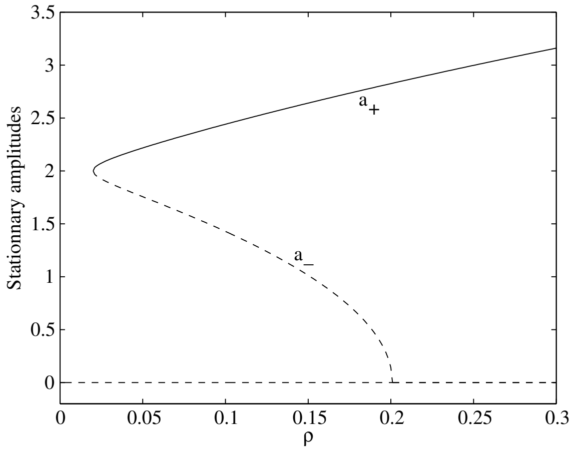

Fig. 2. Stationary amplitudes versus the amplitude ρ of the resonant parametric excitation, for $\alpha = 0.1$, $\beta = 0.1$, $\sigma = 0.01$ and $\omega = 1$. The dashed lines refer to unstable solutions and the continuous line to the stable solution.

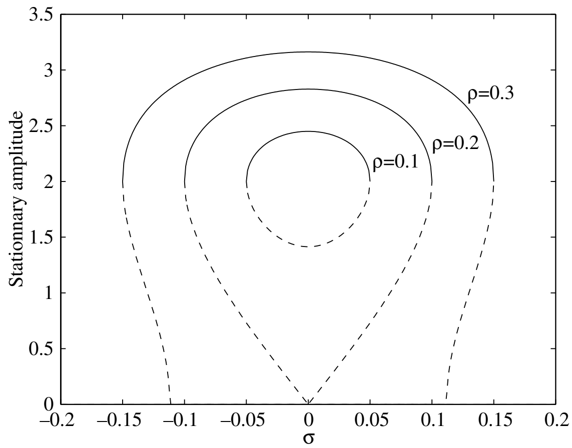

Fig. 3. Stationary amplitudes versus the detuning σ for $\alpha = 0.1$, $\beta = 0.1$, $\omega = 1$. The dashed lines refer to unstable solutions and the continuous line to the stable solution.

resonant excitation ρ. In Fig. 3 the stationary amplitudes $a_{\pm}$ are shown versus the detuning parameter σ for different values of the amplitude ρ of excitation. The maximum amplitudes correspond to the resonance i.e., $\sigma = 0$. In both Figs. 2 and 3 a jump phenomena and a hysterisis cycle are observed in the transition between the trivial and the stable non-trivial solution a_+.

In terms of the original system (1), through the approximation of the solution $x(t)$ given in (16), one can conclude that in the zones *II* and *III* $x(t)$ is a periodic solution with a frequency $\nu/2$. The zone *I* corresponds to a 2-period QP solution, as we will see.

3.1.2. Case $h \neq 0$

When the very slow parametric excitation is present i.e., $h \neq 0$, the non-trivial amplitudes given in (17), are no more constant solutions but periodic with a period $2\pi/\Omega$. This result is obtained through a Fourier series of the square root of the expression given in (17). Through the approximated solution given in (16), the solution of the original system (1) in the zones *II* and *III* is a 2-period-QP solution with fundamental frequencies $\nu/2$ and Ω.

In Figs. 4 and 5 it is respectively shown the numerical solution and the phase space of Eq. (1), for parameter values corresponding to the zone *III*. In Figs. 6 and 7 the power spectra of the signal plotted in Fig. 4 is shown. The numerical

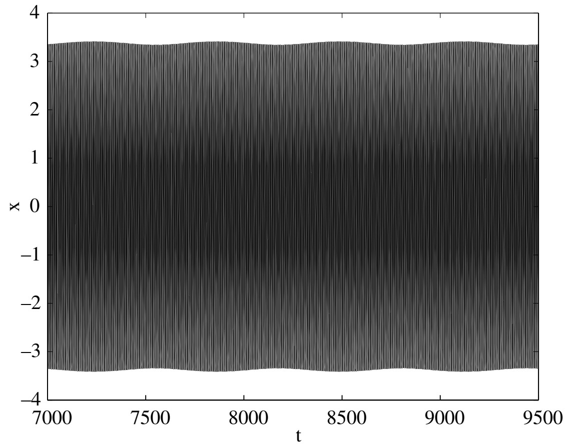

Fig. 4. Time history of Eq. (1) for $\sigma = -0.1$, $\alpha = 0.1$, $\beta = 0.1$, $\rho = 0.4$, $\Omega = 0.01$ and $h = 0.01$.

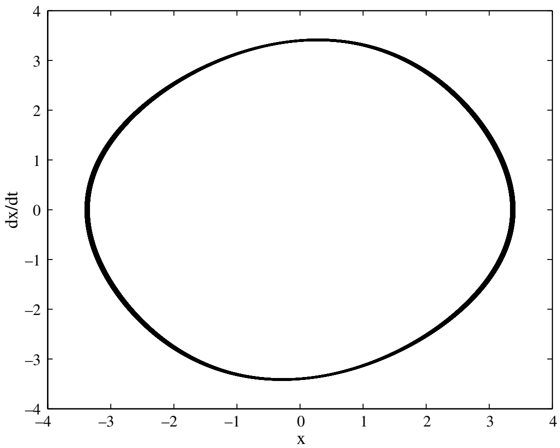

Fig. 5. Phase space of Eq. (1) for $\sigma = -0.1$ for $\alpha = 0.1$, $\beta = 0.1$, $\rho = 0.4$, $\Omega = 0.01$ and $h = 0.01$.

power spectra confirm the analytical predictions concerning the nature of the solution and the fundamental frequencies constructing it.

It is worth noting that the drawn conclusions are valid only when the parameter values are far from the boundaries between the different zones, especially the zones *I* and *II*.

3.2. Limit cycle

In this section, the existence and an approximation of the limit cycle induced by the van der Pol term in the original Eq. (1) is determined.

Expressing the modulation equations of amplitude and phase (14) and (15) in Cartesian form i.e., $u = a\cos(\gamma)$ and $v = -a\sin(\gamma)$ leads to the following equations

$$\frac{du}{dT_1} = \frac{\tilde{\alpha}}{2}u + \left(\frac{\sigma}{2} + \frac{\tilde{\rho}}{4\omega}\right)v - \frac{\tilde{h}}{2\omega}v\cos(\widetilde{\Omega}\tau) - \frac{\tilde{\beta}}{8}(u^2+v^2)u \tag{20}$$

$$\frac{dv}{dT_1} = \frac{\tilde{\alpha}}{2}v + \left(-\frac{\sigma}{2} + \frac{\tilde{\rho}}{4\omega}\right)u + \frac{\tilde{h}}{2\omega}\cos(\widetilde{\Omega}\tau)u - \frac{\tilde{\beta}}{8}(u^2+v^2)v \tag{21}$$

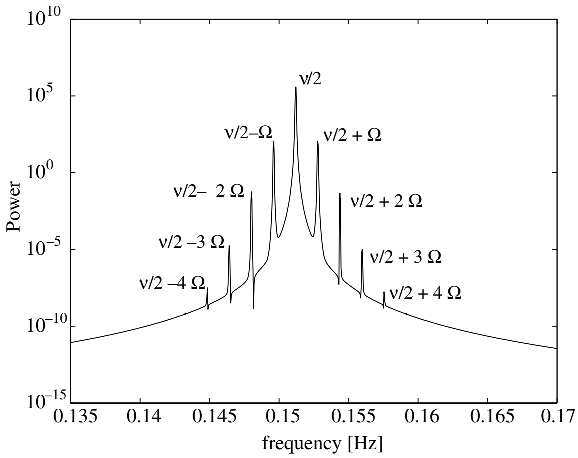

Fig. 6. Power spectrum corresponding to Fig. 4.

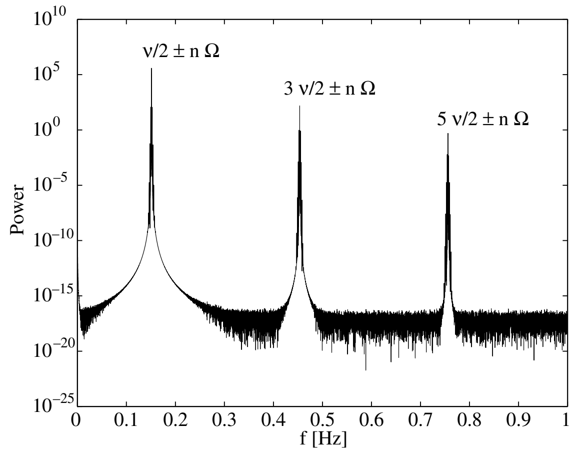

Fig. 7. Magnification of the power spectrum corresponding to Fig. 4.

In order to apply a perturbation method to Eqs. (20) and (21), a perturbation parameter μ in the level of the cubic non-linearities is introduced. Applying the MSM [15] determines an approximate solution in the form

$$u(t) = u_0(t_1, t_2, \tau) + \mu u_1(t_1, t_2, \tau) + \mathcal{O}(\mu^2) \tag{22}$$

where $t_1 = T_1$ is a fast time scale, $t_2 = \mu T_1$ is a slow time scale describing the envelope of the response. In terms of the variable t_n, the time derivative becomes $\mathrm{d}/\mathrm{d}T_1 = (\partial/\partial t_1) + \mu(\partial/\partial t_2) + \mathcal{O}(\mu^2)$.

At different orders of μ the equations of modulations in Cartesian form are written as follows

$$\text{order } \mathcal{O}(1): \frac{\partial u_0}{\partial t_1} = \Omega_1 v_0 \tag{23}$$

$$\frac{\partial v_0}{\partial t_1} = -\Omega_2 u_0 \tag{24}$$

$$\text{order } \mathcal{O}(\mu): \frac{\partial u_1}{\partial t_1} = \frac{\partial u_0}{\partial t_2} + \Omega_1 v_1 + \frac{\tilde{\alpha}}{2} u_0 - \frac{\tilde{\beta}}{8} u_0(u_0^2 + v_0^2) \tag{25}$$

$$\frac{\partial v_1}{\partial t_1} = \frac{\partial v_0}{\partial t_2} - \Omega_2 v_1 + \frac{\tilde{\alpha}}{2} v_0 - \frac{\tilde{\beta}}{8} v_0(u_0^2 + v_0^2) \tag{26}$$

where

$$\Omega_1 = \frac{\sigma}{2} + \frac{\tilde{\rho}}{4\omega} - \frac{\tilde{h}}{2\omega}\cos(\widetilde{\Omega}\tau) \quad (27)$$

$$\Omega_2 = \frac{\sigma}{2} - \frac{\tilde{\rho}}{4\omega} - \frac{\tilde{h}}{2\omega}\cos(\widetilde{\Omega}\tau) \quad (28)$$

$$\Omega^{*2} = \Omega_1\Omega_2 = -H(\tau) \quad (29)$$

From Eq. (29) a necessary condition to have periodic motion is that $H(\tau) < 0$ i.e., the parameters of the resonant parametric excitation, in the absence of the very slow parametric excitation i.e., $h = 0$, should belong to the zone I of Fig. 1.

The solution of the order $\mathcal{O}(\mu^0)$ equations, given in (23) and (24), are expressed as

$$u_0 = B(t_2,\tau)e^{i\Omega^* t_1} + cc \quad (30)$$

$$v_0 = i\frac{\Omega^*}{\Omega_1}B(t_2,\tau)e^{i\Omega^* t_1} + cc \quad (31)$$

where cc is the complex conjugate of the preceding terms. The condition of elimination of secular terms from Eqs. (25) and (26) leads to

$$2\frac{\partial B}{\partial t_2} = \tilde{\alpha}B - \frac{\tilde{\beta}}{2}\left(1 + \frac{\Omega_2}{\Omega_1}\right)B^2\overline{B} \quad (32)$$

Setting $B = (b/2)\exp(i\phi)$ in Eq. (32), the equations of modulations of amplitude b and phase ϕ are given by

$$\frac{\partial b}{\partial t_2} = \frac{\tilde{\alpha}}{2}b - \frac{\tilde{\beta}}{16}\left(1 + \frac{\Omega_2}{\Omega_1}\right)b^3 \quad (33)$$

$$b\left(\frac{\partial\phi}{\partial t_2}\right) = 0 \quad (34)$$

Eq. (33) has two stationary solutions: an unstable trivial solution and a stable non-trivial solution corresponding to the amplitude of the limit cycle given by

$$b^2(\tau) = \frac{8\tilde{\alpha}}{\tilde{\beta}\left(1 + \frac{\Omega_2}{\Omega_1}\right)} \quad (35)$$

The phase ϕ is a constant. It is clear that when $h = 0$ the amplitude b of the limit cycle is a constant. It is worth noting that the amplitude b depends mainly on the parameters of the self-excitation i.e., α and β.

Up to the first order an approximation of the solutions of the modulation equations (14) and (15) is defined as

$$a = b(\tau)\sqrt{\cos^2(\Omega^* T_1 + \phi) + \left(\frac{\Omega^*}{\Omega_1}\right)^2 * \sin^2(\Omega^* T_1 + \phi)} \quad (36)$$

$$\gamma = \arctan\left(\frac{\Omega^*}{\Omega_1}\tan(\Omega^* T_1 + \phi)\right) \quad (37)$$

In the region I far from the boundary with the zone II, for $h = 0$ and $\rho = 0$ Eq. (1) has a limit cycle due to the self-excitation. When $h = 0$ and $\rho \neq 0$ in the zone I Eq. (1) has a 2-period QP solution with $\nu/2$ and $2\Omega^*$ as fundamental frequencies.

When $\rho \neq 0$ and $h \neq 0$, in the zone I far from the boundaries with zone II, the solution of Eq. (1) is 3-period-QP solution with $(\nu/2)$, Ω and $2\Omega_0^*$ as fundamental frequencies. Here Ω_0^* corresponds to Ω^* given in (29) for $h = 0$.

In Figs. 8 and 9, the numerical time series and the phase space of Eq. (1) are shown. The parameter's values are belonging to the zone I of Fig. 1 and are far from the boundaries. In Fig. 10 the power spectrum of the signal and its fundamental frequencies are indicated.

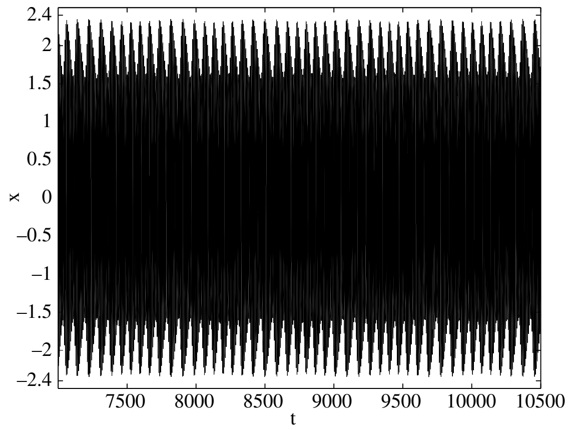

Fig. 8. Time history of Eq. (1) for $\sigma = -0.1$, $\alpha = 0.1$, $\beta = 0.1$, $\rho = 0.1$, $\Omega = 0.01$ and $h = 0.01$.

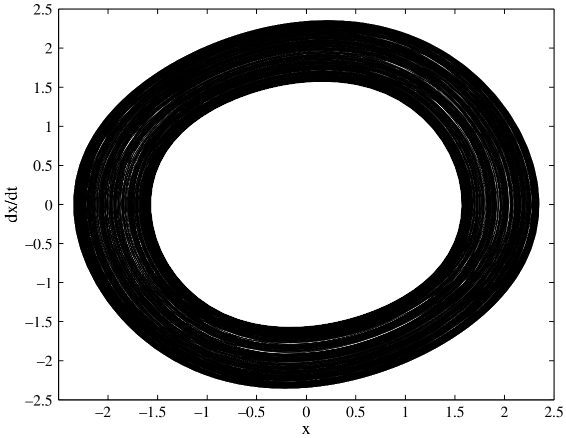

Fig. 9. Phase space of Eq. (1) for $\sigma = -0.1$, $\alpha = 0.1$, $\beta = 0.1$, $\rho = 0.1$, $\Omega = 0.01$ and $h = 0.01$.

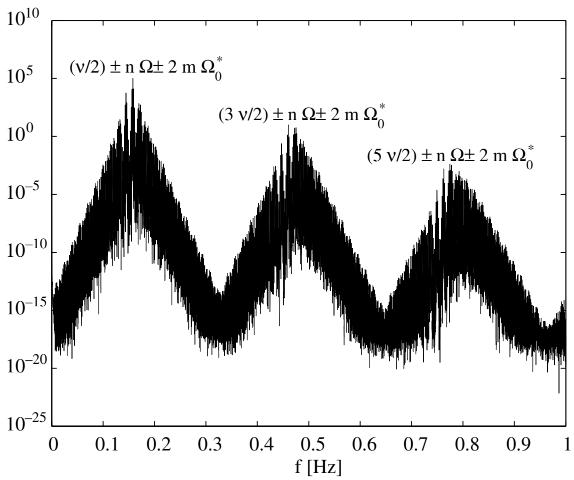

Fig. 10. Power spectrum corresponding to the signal in Fig. 8.

4. Periodic bursters

Through the expressions of the quasi-static solutions of the modulations equations (17), (18) and (36), (37), in the presence of the very slow parametric excitation, it is seen that periodic bursters depend on the very slow frequency Ω. Therefore in a very long period $2\pi/\Omega$ a given quasi-static solution may change its stability or even disappear e.g., the limit cycle induced by self-excitation. The changes in the nature of a given solution are more likely for a choice of the parameters, in the absence of the very slow excitation, putting the system in the vicinity of the boundary between the zones *I* and *II*. Hence, this crossing of these boundaries during one period of the very slow dynamics leads to the appearance of periodic bursters. In Fig. 11, we re-plot Fig. 1 with the effect of the very slow excitation included. In the grey zones a given solution changes its stability or ceases to exist in a part of the period of the very slow dynamics. It can be conjectured that these grey regions are the regions of existence of periodic bursters. In the grey region between the zones *I* and *II*, the periodic burster solution is a heteroclinic connection between the 2-period-QP solution and the 3-period-QP solution. The grey region between the zones *II* and *III* does not involve any change for the stable 2-period QP solution; it involves only the disappearance of the unstable 2-period QP solution. In the zone *I* the solution of the system (1) is 3-period-QP. In zones *II* and *III* the solution is 2-period-QP. In Figs. 12–14 a periodic burster solution

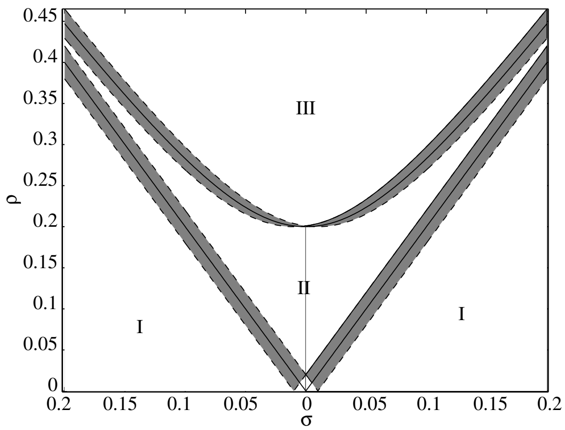

Fig. 11. Chart of behaviors for $\alpha = 0.1$, $\beta = 0.1$, $\Omega = 0.01$ and $h = 0.01$. In the zone *I* coexistence of an unstable trivial solution and a 3-period-QP solution. In the zone *II* coexistence of the unstable trivial fixed point and two 2-period-QP solutions, one among them is stable. In zone *III* only the stable 2-period-QP solution and the unstable trivial solution coexist. Grey regions correspond to periodic bursters.

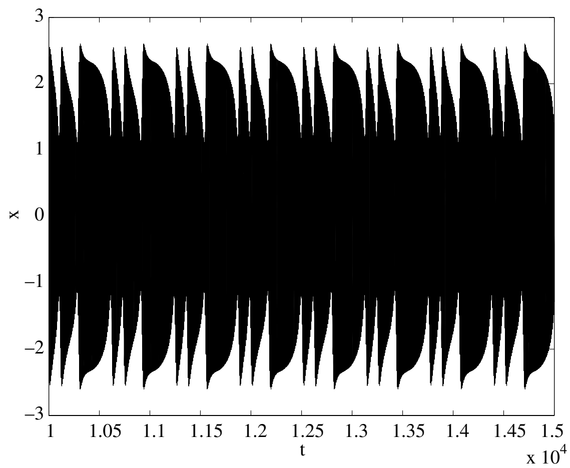

Fig. 12. Periodic burster solution involving 2-period QP solution and 3-period QP solution. $\rho = 0.185$, $\nu = 1.9$, $\alpha = 0.1$, $\beta = 0.1$, $\Omega = 0.01$ and $h = 0.01$.

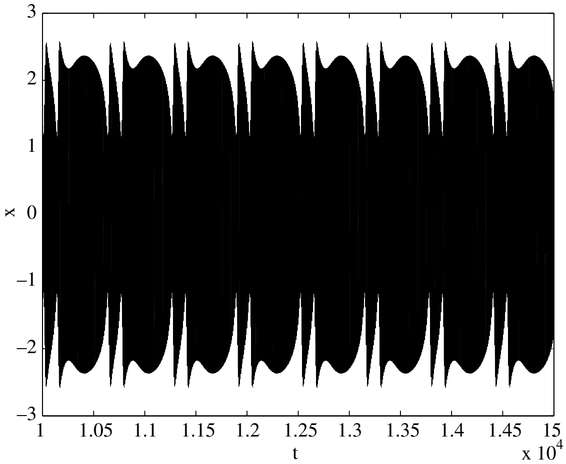

Fig. 13. Periodic burster solution involving 2-period QP solution and 3-period QP solution. $\rho = 0.19$, $\nu = 1.9$, $\alpha = 0.1$, $\beta = 0.1$, $\Omega = 0.01$ and $h = 0.01$.

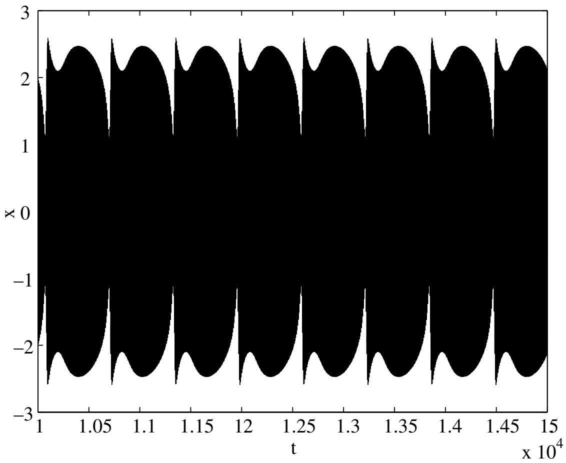

Fig. 14. Periodic burster solution involving 2-period QP solution and 3-period QP solution. $\rho = 0.2$, $\nu = 1.9$, $\alpha = 0.1$, $\beta = 0.1$, $\Omega = 0.01$ and $h = 0.01$.

involving the 3-period-QP and 2-period-QP solutions is plotted, for the parameters values belonging to the grey zone in Fig. 11. In these figures the amplitude of the resonant excitation ρ is varied, such that the burster is spending less and less time in the 3-period QP solution i.e., it approaches more and more the boundary with zone *II*.

5. Conclusion

In this paper the regular dynamics of a van der Pol–Mathieu equation subject to a quasi-periodic parametric excitation has been studied. The latter consists of a resonant frequency with the proper frequency of the oscillator and a very slow frequency. An averaging of the fast dynamics enables the determination of the quasi-static solutions which form the slow manifold. A second perturbation method was performed to approximate the self-excited contribution on the modulations equations of amplitude and phase. Different charts of regular behaviors are determined, including 2-period-quasi-periodic solution, 3-period-quasi-periodic solution, and periodic bursters relating them. The periodic bursters are due to the very slow perturbation which enables the solutions to cross the boundaries between different behaviors during one period of it.

As a continuation of this study the different bifurcations of the determined regular solutions and the roots to chaos and to strange non-chaotic attractors will be investigated.

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