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# 4-SUBHARMONIC BIFURCATION AND HOMOCLINIC TRANSITION NEAR RESONANCE POINT IN NONLINEAR PARAMETRIC OSCILLATOR

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## 1. Introduction

Given the Mathieu-type differential equation

$$\ddot{x} + \alpha\dot{x} + \beta x\,\dot{x} + \omega_0^2\,(1 + h\cos\omega t\,)\,x = cx^2 \tag{1}$$

where $\alpha$, $\beta$, $\omega_o$, h, and c are real constants, $\omega$ a real positive and $\mu$ = -$\alpha$, $\omega_o$). Periodic solutions for (1) can be found by using either the stroboscopic method which associates the Poincaré map T, or by application of formal asymptotic method (eg. see [14]). A fixed point of T (corresponding to a $2\pi/\omega$–periodical solution of (1)) with complex eigenvalues s and $\bar{s}$ present a Poincaré-Hopf bifurcation of resonance p/q when s = exp $(2i\pi p/q)$, with p and q relatively prime. In the $\mu$-parameter space, we shall call $P_{p/q}$ the corresponding point for that bifurcation.

In [3] we have constructed numerically the 4-subharmonic solutions for (1) near point of resonance $P_4$ (p = 1, q = 4) and we have shown ( as in [1] and [11] ) that there exists a horn $K_4$ in the $\mu$-parameter space corresponding to the existence of these solutions. On the other hand, we have shown (numerically in [3]) the coexistence of 4-subharmonics and closed invariant curve in horn $K_4$ near $P_4$ ( see also [2] for external excitation case ). The boundaries $N_4^1$ and $N_4^2$ of this horn are defined by the saddle-node bifurcation locus of 4-subharmonics produced at $P_4$ ( Fig. 1).

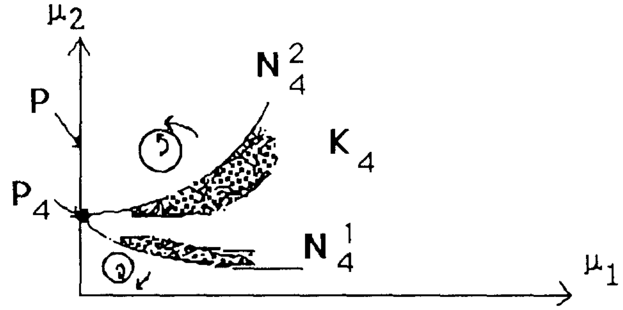


Fig. 1. [pattern] : coexistence region of 4-subharmonics and invariant curve,
P : Hopf bifurcation locus, $P_4$ : resonace point of order 4,
$N_4^{1,2}$: saddle node bifurcation locus.

In order to approximate $N_4^{1,2}$ theoretically, several authors ([8], [9]) employ the Melnikov function $M^4(t_0)$ for the 4-subharmonic bifurcations and in particular the function's approximation to the first order $M_1^4(t_0)$. This function, which is a variant of the Melnikov function for homoclinic bifurcations ([13]) $M_1(t_0)$ ( where $M_1(t_0) = \lim M_1^q(t_0)$ when $q \rightarrow \infty$ ) does not allow us to approximate the 4-subharmonics.

The surpose of this paper is to construct near resonance point of order 4 analytical approximations of the 4-subharmonics, the saddle-node bifurcation locus of 4-subharmonics, and the homoclinic transition locus of principal saddle for (1). In particular, we determine the region in phenomelogical parameters space in which the ordinary principal saddle (or non-homoclinic saddle) is in the presence of two asymptotically stable solutions.

## 2. Saddle-node bifurcation locus of 4-subharmonics

One of methods adapted to construct $N_4^{1,2}$ is the Bogoliubov-Mitropolsky method ([7]). However, this method which was developed to order 2 in $\varepsilon$ and which allows to study the fundamental or 2-subharmonic solutions, creates some difficulty for the study of q-resonance when $q \geq 4$. Indeed, the presence of multiple solutions near $P_q$ requires a development of this method to order 3. We therefore suggest expanding this order within a more general case and applying it to (1) for $q = 4$.

In order to construct the solutions of (1) in the neighbourhood of $P_q$, we impose: $\omega_0^2 = (\omega / q)^2 + \Delta\omega$, and then the dependance of the parameters on $\varepsilon$ will be selected accordingly ([10])

$$h = \varepsilon\tilde{h}, \quad \beta = \varepsilon\tilde{\beta}, \quad c = \varepsilon\tilde{c}, \quad \alpha = \varepsilon^2\tilde{\alpha}, \quad \Delta\omega = \varepsilon^2\widetilde{\Delta\omega} \tag{2}$$

where $\varepsilon$ is small parameter. By substituting (2) into (1) we obtain

$$\ddot{x} + \left(\frac{p\omega}{q}\right)^2 x = \varepsilon f(x, \dot{x}, \omega t) + \varepsilon^2 g(x, \dot{x}, \omega t) + \varepsilon^3 h(x, \dot{x}, \omega t) \tag{3}$$

where

$$f = \tilde{c}x^2 - \tilde{\beta}x\dot{x} - \tilde{h}\left(\frac{p\omega}{q}\right)^2 x \cos\omega t, \qquad g = -\widetilde{\Delta\omega}x - \tilde{\alpha}\dot{x}, \qquad h = -\widetilde{\Delta\omega}\,\tilde{h}x \cos\omega t.$$

As in [7], we find the solutions of (3) in the form

$$x = a\cos\psi + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 \ldots, \qquad \psi = p\omega t/q + \theta \tag{4}$$

where each $u_i$ $(a, \psi, \omega t)$ is $2\pi$-periodic function in $\psi$ and $\omega t$. The amplitude a (t) and the phase $\theta(t)$ are defined by the system

$$\frac{da}{dt} = \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 \ldots,$$
$$\frac{d\theta}{dt} = \varepsilon B_1 + \varepsilon^2 B_2 + \varepsilon^3 B_3 \ldots \tag{5}$$

where $A_i(a,\theta)$, $B_i(a,\theta)$ are $2\pi$-periodic functions in $\theta$.

Substituting (4) and (5) into (3), expanding, and equating coefficients of like powers of $\varepsilon$, we have the first three terms

$$
\left\{
\begin{aligned}
&\frac{\partial^2 u_1}{\partial t^2} + \left(\frac{p\omega}{q}\right)^2 u_1 = f + \frac{2p\omega}{q} A_1 \sin\psi + 2a\frac{p\omega}{q} B_1 \cos\psi, \\
&\frac{\partial^2 u_2}{\partial t^2} + \left(\frac{p\omega}{q}\right)^2 u_2 = \left(\frac{2p\omega}{q} A_2 + a\frac{\partial B_1}{\partial a} A_1 + a\frac{\partial B_1}{\partial \theta} B_1 + 2A_1B_1\right)\sin\psi \\
&+ \left(2a\frac{p\omega}{q} B_2 - \frac{\partial A_1}{\partial a} A_1 - \frac{\partial A_1}{\partial \theta} B_1 + aB_1^2\right)\cos\psi \\
&+ \frac{\partial f}{\partial x} u_1 - 2A_1 \frac{\partial^2 u_1}{\partial a \partial t} - 2B_1 \frac{\partial^2 u_1}{\partial \theta \partial t} + \left(A_1\cos\psi - aB_1\sin\psi + \frac{\partial u_1}{\partial t}\right)\frac{\partial f}{\partial \dot{x}} + g, \\
&\frac{\partial^2 u_3}{\partial t^2} + \left(\frac{p\omega}{q}\right)^2 u_3 = \left(\frac{2ap\omega}{q} B_3 - \frac{\partial A_2}{\partial a} A_1 - \frac{\partial A_1}{\partial a} A_2 - \frac{\partial A_2}{\partial \theta} B_1 - \frac{\partial A_1}{\partial \theta} B_2 + 2aB_1B_2\right)\cos\psi \\
&+ \left(2\frac{p\omega}{q} A_3 + a\frac{\partial B_2}{\partial a} A_1 + a\frac{\partial B_1}{\partial a} A_2 + a\frac{\partial B_2}{\partial \theta} B_1 + a\frac{\partial B_1}{\partial \theta} B_2 + 2(A_1B_2 + A_2B_1)\right)\sin\psi \\
&- \frac{\partial u_1}{\partial a}\left(A_1\frac{\partial A_1}{\partial a} + B_1\frac{\partial A_1}{\partial \theta}\right) - \frac{\partial u_1}{\partial \theta}\left(A_1\frac{\partial B_1}{\partial a} + B_1\frac{\partial B_1}{\partial \theta}\right) - A_1^2\frac{\partial^2 u_1}{\partial a^2} - B_1^2\frac{\partial^2 u_1}{\partial \theta^2} - 2A_2\frac{\partial^2 u_1}{\partial a \partial t} \\
&- 2A_1B_1\frac{\partial^2 u_1}{\partial a \partial \theta} - 2B_2\frac{\partial^2 u_1}{\partial \theta \partial t} - 2A_1\frac{\partial^2 u_2}{\partial a \partial t} - 2B_1\frac{\partial^2 u_2}{\partial \theta \partial t} + u_1\frac{\partial g}{\partial x} + u_2\frac{\partial f}{\partial x} + h \\
&+ \left(\frac{\partial u_1}{\partial t} + A_1\cos\psi - aB_1\sin\psi\right)\frac{\partial g}{\partial \dot{x}} + \left(\frac{\partial u_2}{\partial t} + A_2\cos\psi - aB_2\sin\psi + A_1\frac{\partial u_1}{\partial a} + B_1\frac{\partial u_1}{\partial \theta}\right)\frac{\partial f}{\partial \dot{x}} \\
&+ \frac{u_1^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u_1}{\partial t} + A_1\cos\psi - aB_1\sin\psi\right)^2\frac{\partial^2 f}{\partial \dot{x}^2} + u_1\left(\frac{\partial u_1}{\partial t} + A_1\cos\psi - aB_1\sin\psi\right)\frac{\partial^2 f}{\partial x \partial \dot{x}}
\end{aligned}
\right. \qquad (6a,b,c)
$$

where all the derivatives of f and g are evaluated for $x = a\cos\psi$ and $\dot{x} = -(ap\omega/q)\sin\psi$. In the special case: $g = 0$ and $h = 0$ we find the first two terms (6,a,b) obtained in [7] (p. 217-218).

In order to determine functions $u_i$, $A_i$ and $B_i$, we take $u_i = 0$ as the solution without a second member and suppose the absence of secular terms in $u_i$. The periodic solutions for (3), which correspond to the stationary regimes of (5), are the roots of the algebraic system

$$\varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \ldots = 0, \qquad \varepsilon B_1 + \varepsilon^2 B_2 + \varepsilon^3 B_3 + \ldots = 0. \qquad (7)$$

The application of the asymptotic method to equation (3) gives the first order (for p/q=1/4)

$$x = a\cos\left(\frac{\omega t}{4} + \theta\right), \qquad \frac{da}{dt} = 0, \qquad \frac{d\theta}{dt} = 0 \tag{8}$$

and then the second order

$$\begin{cases} x = a\cos(\omega t/4 + \theta) + u_1 \\ \dfrac{da}{dt} = -\dfrac{\alpha a}{2} - \dfrac{2\beta c}{\omega^2} a^3, \qquad \dfrac{d\theta}{dt} = \dfrac{2\Delta\omega}{\omega} + \dfrac{h^2\omega}{192} - \left(\dfrac{\beta^2}{6\omega} + \dfrac{80c^2}{3\omega^3}\right) a^2, \\ u_1 = \dfrac{8ca^2}{\omega^2} - \dfrac{8ca^2}{3\omega^2}\cos\left(\dfrac{\omega t}{2} + 2\theta\right) - \dfrac{2\beta a^2}{3\omega}\sin\left(\dfrac{\omega t}{2} + 2\theta\right) \\ \quad + \dfrac{ha}{16}\left(\dfrac{1}{3}\cos\left(\dfrac{5\omega t}{4} + \theta\right) + \cos\left(\dfrac{3\omega t}{4} - \theta\right)\right). \end{cases} \tag{9}$$

The stationary values of a are given by

$$a^2 = -\frac{\alpha\omega^2\delta}{4\beta c} \quad (\delta = 0 \text{ or } 1), \tag{10a}$$

$$a^2 = \frac{6\omega^3}{160c^2 + \beta^2\omega^2}\left(\frac{2\Delta\omega}{\omega} + \frac{h^2\omega}{192}\right). \tag{10b}$$

In consequence, it follows on the one hand that $a = 0$, which corresponds to the solution $x(t) = 0$ ( stable for $\alpha > 0$ and unstable for $\alpha < 0$ ) and on the other hand that:

$$a = \sqrt{-\alpha\omega^2/4\beta c} \tag{11a}$$

$$\omega_0^2 = \left(\frac{\omega}{4}\right)^2 - \frac{h^2\omega^2}{384} - \left(\frac{160c^2 + \beta^2\omega^2}{24\,\beta c\omega}\right)\alpha \tag{11b}$$

thus establishing an approximation of the stable quasi-periodic solution C ( for $\alpha < 0$ ) and its bifurcation locus B (Fig. 2).

From (6c) we obtain the 3rd order approximation of the 4-subharmonics

$$
\left\{
\begin{aligned}
&x = a\cos(\omega t/4+\theta) + u_1 + u_2 \\
&\frac{da}{dt} = -\frac{\alpha a}{2} - \frac{2\beta c a^3}{\omega^2} + ha^3\left(\left(\frac{20c^2}{9\omega^3} - \frac{\beta^2}{72\omega}\right)\sin 4\theta - \frac{\beta c}{9\omega^2}\cos 4\theta\right) \\
&\frac{d\theta}{dt} = \frac{2\Delta\omega}{\omega} + \frac{h^2\omega}{192} - \left(\frac{\beta^2}{6\omega} + \frac{80c^2}{3\omega^3}\right)a^2 + ha^2\left(\left(\frac{20c^2}{9\omega^3} - \frac{\beta^2}{72\omega}\right)\cos 4\theta + \frac{\beta c}{9\omega^2}\sin 4\theta\right) \\
&u_2 = \left(\frac{16c^2}{3\omega^4} - \frac{\beta^2}{2\omega^2}\right)a^3\cos 3\psi + \frac{10\beta c a^3}{3\omega^3}\sin 3\psi + \frac{20hca^2}{45\omega^2}\cos 4(\psi-\theta) - \frac{\beta h a^2}{20\omega}\sin 4(\psi-\theta) \\
&\quad - \frac{7hca^2}{9\omega^2}\cos 2(\psi-2\theta) + \frac{h\beta a^2}{36\omega}\sin 2(\psi-2\theta) - \frac{hca^2}{21\omega^2}\cos 2(3\psi-2\theta) \\
&\quad - \frac{\beta h a^2}{60\omega}\sin 2(3\psi-2\theta) + \frac{h^2 a}{7680}\cos(9\psi-8\theta) + \frac{h^2 a}{1536}\sin(7\psi-8\theta).
\end{aligned}
\right. \tag{12}
$$

The stationary values of a and $\theta$ are given by (12b,c). After elimination of $\theta$ we obtain a second order equation in $a^2$, which gives real solutions when its discriminant is positive or nul. This condition is given by

$$
\left\{
\begin{aligned}
&(\omega/4)^2 + \Delta_1 \le \omega_0^2 \le (\omega/4)^2 + \Delta_2 \\
&\Delta_{1,2} = ((8D_0B_0\alpha - 16C_0(A+B_0^2))\omega \pm \sqrt{-64A\omega^2\alpha^2(D_1^2+D_2^2)}\,)\,/\,32(A+B_0^2), \\
&A = D_1^2 + D_2^2 - D_0^2 - B_0^2, \qquad D_0 = 2\beta c/\omega^2, \qquad B_0 = \beta^2/6\omega + 80c^2/3\omega^3, \\
&C_0 = h^2\omega/192, \qquad D_1 = 20hc^2/9\omega^3 - \beta^2 h/72\omega, \qquad D_2 = \beta hc/9\omega^2.
\end{aligned}
\right. \tag{13}
$$

For parameter values such as $A \le 0$, equation (13) defines the existence region for 4-subharmonic which appears as a saddle-node bifurcation on the curves $N_4^{1,2}$ given by

$$
\omega_0^2 = (\omega/4)^2 + \Delta_1 \quad \text{and} \quad \omega_0^2 = (\omega/4)^2 + \Delta_2 \tag{14}
$$

For the fixed parameter values: $\omega = 2$, $\beta = .2$ , $c = 1$ and $h = .1$, the locus $N_4^{1,2}$ are represented by a dotted line in Fig. 2 and compared to the locus (crossed line) obtained numerically in [3]. Furthermore, the coordinates of point $P_4$ given by both methods are: $\alpha = 0$, $\omega_0 = .498896$. The curve B, illustrated in Fig. 2 (solid line), approximates the saddle-loop bifurcation locus, which establishes the only possible transition to describes the destabilisation of C (see also [3], [2], [6], [4] for similar results). The locus B is compared to a single numerical value (single point) obtained numerically in [3].

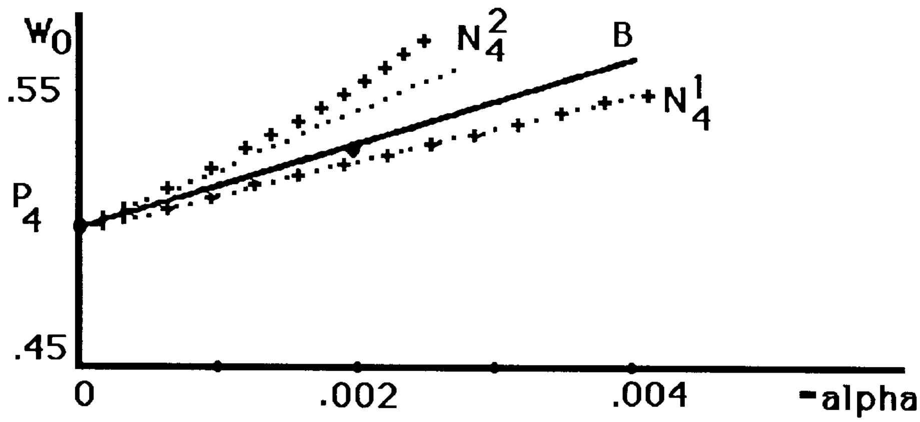


Fig. 2. Bifurcation locus: $N_4^{1,2}$( ...anal. approximation, +++ numer. integration),
B (— anal. approximation, • numer. integration).

These results give: In the domain delimited by curves B and $N_4^1$, the stable quasi-periodic solution C and the 4-subharmonic coexist near $P_4$.

In fact the slope of a curve B given by (11b) is included between those of curves $N_4^1$ and $N_4^2$ given by (14) (see Fig. 2). Note that theorems available at present ([15],[12]) specify the existence of an invariant curve C (of T) only on the exterior of $K_4$.

## 3. Homoclinic transition

Using the Melnikov method [13] to study equation (1), we obtain the homoclinic Melnikov function

$$M_1(t_0) = 6\omega_0^6 h \left(\frac{I_1(\omega_0) - 6I_2(\omega_0)}{c^2}\right) \sin \omega t_0 - \frac{216\,\omega_0^7 \beta K}{c^3} + 36\omega_0^5 \left(\frac{\alpha}{c^2} + \frac{\beta\omega_0^2}{c^3}\right) J \qquad (15)$$

where

$$I_1 = \int_{-\infty}^{+\infty} \left[\frac{\exp\tau\,(1-\exp\tau)\sin(\omega\tau/\omega_0)}{(1+\exp\tau)^3}\right] d\tau, \quad I_2 = \int_{-\infty}^{+\infty} \left[\frac{\exp 2\tau\,(1-\exp\tau)\sin(\omega\tau/\omega_0)}{(1+\exp\tau)^5}\right] d\tau$$

$$J = \int_{-\infty}^{+\infty} \left[\exp 2\tau\,(1-\exp\tau)^2/(1+\exp\tau)^6\right] d\tau, \quad K = \int_{-\infty}^{+\infty} \left[\exp 3\tau\,(1-\exp\tau)^2/(1+\exp\tau)^8\right] d\tau$$

The first two integrals in (15) can be evaluated by the method of residues and the two last ones directly. we obtain

$$J = \frac{1}{30}, \quad K = \frac{1}{210}, \quad I_1 = \frac{-4\pi\,(\omega/\omega_0)^2}{\mathrm{sh}\,(\pi\omega/\omega_0)}, \quad I_2 = \frac{-\pi\,((\omega/\omega_0)^2+(\omega/\omega_0)^4)}{3\,\mathrm{sh}\,(\pi\omega/\omega_0)} \tag{16}$$

For the fixed parameter values: $\omega = 2$, $\beta = .2$, $c = 1$ and $h = .1$, the approximation $H_1$, $H_2$ of the homoclinic transition locus in the parameter space $\mu = (-\alpha, \omega_0)$ is given by calculating the quadratic zero of $M_1(t_0)$, we have

$$\alpha = -\frac{.2}{7}\,\omega_0^2 \pm \frac{8\pi}{2\omega_0\,\mathrm{sh}\,(2\pi/\omega_0)}\,(4/\omega_0^2 - 1) \tag{17}$$

In Fig. 3 we show the bifurcation curves $H_{1,2}$ and $N_4^{1,2}$. In the region 1 there exist transverse homoclinic intersection points of the stable and unstable manifolds of the principal saddle $(\omega_0^2/c, 0)$. In the region 2 the 4-subharmonics and the quasi-periodic solution C (invariant closed curve) may exist in the presence of a regular saddle ( or non-homoclinic saddle).

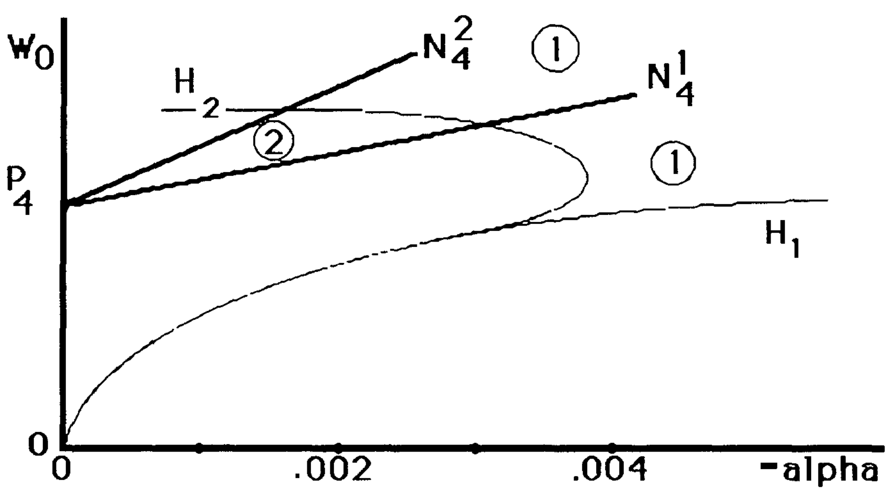


Fig. 3. $H_{1,2}$ : Homoclinic transition locus of equation (1),
$N_4^{1,2}$: saddle-node bifurcation locus of 4-subharmonics.

## 4. Conclusions

The numerical constructions of the periodic solutions and their bifurcation locus in parameter space near a resonance point of order 4 require very extensive treatment using several algorithms simultaneously (fixed point algorithms [2], Lattès method [5]). On the other hand, a 3rd order asymptotic method gives near $P_4$ such approximations in the whole phenomelogical parameter space directly from the system itself. This 3rd order expansion may also be used to examine the 4-resonance for other oscillators representing a degenerate Poincaré-Hopf bifurcation of 4-resonance and allows us, in general, to obtain better approximations for the other resonances.

The coexistence of two asymptotically stable solutions near $P_4$ may to occur in the region of parameter space where the principal saddle is not homoclinic.

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